

A MOTIVIC GROTHENDIECK-TEICHMÜLLER GROUP

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ABSTRACT. This paper proves the Beilinson-Soulé vanishing conjecture for motives attached to the moduli spaces of curves of genus 0 with n marked points, $\mathcal{M}_{0,n}$. As part of the proof, it is also proved that these motives are mixed Tate. As a consequence of Levine's work, one obtains then well defined categories of mixed Tate motives over the moduli spaces of curves $\mathcal{M}_{0,n}$. It is shown that morphisms between $\mathcal{M}_{0,n}$'s forgetting marked points and embedding as boundary components induce functors between those categories and how tangential bases points fit in these functorialities.

Tannakian formalism attaches groups to these categories and morphisms reflecting the functorialities leading to the definition of a motivic Grothendieck-Teichmüller group.

Proofs of the above properties rely on the geometry of the tower of the $\mathcal{M}_{0,n}$. This allows us to treat the general case of motives over $\mathrm{Spec}(\mathbb{Z})$ with \mathbb{Z} coefficients working in Spitzweck's category of motives. From there, passing to \mathbb{Q} -coefficients we deal with the classical tannakian formalism and explain how working over $\mathrm{Spec}(\mathbb{Q})$ allows a more concrete description of the tannakian group.

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1. INTRODUCTION

In [Lev10], M. Levine considers a smooth quasi-projective variety X over a number field \mathbb{F} . He shows that when the motive of X in $\mathrm{DM}_{/\mathbb{F},\mathbb{Q}}(\mathrm{Spec}(\mathbb{F}))$ is mixed Tate and satisfies the Beilinson-Soulé vanishing properties, one has a well defined tannakian category of mixed Tate motives $\mathrm{MTM}_{/\mathbb{F},\mathbb{Q}}(X)$ whose tannakian Hopf algebra H_X is built out of a complex of algebraic cycles computing the higher Chow groups. Moreover M. Levine proves that the tannakian group $G_X = \mathrm{Spec}(H_X)$ fits in a short exact sequence

$$1 \longrightarrow G_{X,geom} \longrightarrow G_X \longrightarrow G_{\mathrm{Spec}(\mathbb{F})} \longrightarrow 1$$

where $G_{X,geom}$ can be identified with Deligne-Goncharov motivic fundamental group $\pi_1^{mot}(X, x)$ after a choice of a (tangential) base point $x \in X(\mathbb{F})$.

The above exact sequence admits a Lie coalgebra counterpart

$$0 \longrightarrow L_{\mathrm{Spec}(\mathbb{F})}^c \longrightarrow L_X^c \longrightarrow L_{X,geom}^c \longrightarrow 0$$

by considering the set of indecomposable elements of H_X . In [Sou14], the author shows how explicit algebraic cycles, built in [Sou12], describe the coaction of $L_{\mathrm{Spec}(\mathbb{F})}^c$ on $L_{X,geom}^c$ in the case where

$$X = \mathcal{M}_{0,4} \simeq \mathbb{P}^1 \setminus \{0, 1, \infty\}.$$

In order to generalize this work to any $\mathcal{M}_{0,n}$, the moduli space of curves in genus 0 with n marked points, the first step consists in showing that the moduli spaces $\mathcal{M}_{0,n}$ satisfy the Beilinson-Soulé vanishing conjecture.

However, if working over $\mathrm{Spec}(\mathbb{F})$ allows M. Levine to relate H_n to a cycle complex computing motivic cohomology, the moduli space of curves are well-defined over $\mathrm{Spec}(\mathbb{Z})$. Therefore there is no reason to restrict ourselves to $\mathrm{Spec}(\mathbb{F})$ when considering only the Beilinson-Soulé vanishing property as one can simply work in Cisinski-Dégliše framework [CD09]. Even more generally, the Beilinson-Soulé vanishing property and the mixed Tate property are true in M. Spitzweck framework [Spi13] of motives over $\mathrm{Spec}(\mathbb{Z})$ with \mathbb{Z} coefficients as proved at Theorem 3.8.

From Theorem 3.8 giving the Beilinson-Soulé vanishing property for the moduli spaces of curves $\mathcal{M}_{0,n}$, we deduce from M. Spitzweck's work [Spi13, Spi10] that there exists a well defined triangulated category $\mathrm{DMT}_{/\mathrm{Spec}(\mathbb{Z}),\mathbb{Z}}^{gm}(\mathcal{M}_{0,n})$ of mixed Tate motives over the $\mathcal{M}_{0,n}$ (Theorem 4.9). There are natural morphisms making the moduli spaces $\mathcal{M}_{0,n}$ “into a tower”. These morphisms are given by forgetting some marked points and by embedding of $\overline{\mathcal{M}}_{0,n_1} \times \mathcal{M}_{0,n_2}$ as codimension 1 boundary component of $\overline{\mathcal{M}}_{0,n_1+n_2-2}$ on the Deligne-Mumford compactified tower. These morphisms induce functors between the categories $\mathrm{DMT}_{/\mathrm{Spec}(\mathbb{Z}),\mathbb{Z}}^{gm}(\mathcal{M}_{0,n})$ and morphisms between the corresponding tannakian groups when working with \mathbb{Q} coefficients. This leads to the definition of a motivic Grothendieck-Teichmüller group at Definition 5.1.

The structure of the paper is as follows:

- Section 2 reviews the framework of motivic \mathbb{P}^1 spectra and the stable motivic homotopy category $\mathrm{SH}(S)$. It presents shortly M. Spitzweck's triangulated category of mixed motives over S and reviews some of its properties: Gysin/localization triangle, projective bundle formula and blow-up formula. These are derived from Déglise's work [Dég08] because M. Spitzweck construction relies on an oriented E_∞ -ring spectrum.
- Section 3 reviews the geometry of the $\mathcal{M}_{0,n}$ and their Deligne-Mumford compactification $\overline{\mathcal{M}}_{0,n}$. It proves that the triviality of normal bundle of D_0 in $\mathcal{M}_{0,n} \cup D_0$ for any open codimension 1 stratum of $\overline{\mathcal{M}}_{0,n}$. Then it proves that the motives of $\overline{\mathcal{M}}_{0,n}$ are mixed Tate over $\mathrm{Spec}(\mathbb{Z})$ and satisfies the

Beilinson-Soulé vanishing property. Then, it proves that the same holds for the open moduli spaces $\mathcal{M}_{0,n}$.

- Section 4 begins by reviewing the construction of limits motives as developed in [Spi01, Spi05] and in [Ayo07b] and the case of motivic tangential base points. Then it shows how limits motives applied to the moduli space of curves $\mathcal{M}_{0,n}$ and an open codimension 1 stratum D_0 lead to natural functor between $\mathrm{DMT}_{/\mathrm{Spec}(\mathbb{Z}),\mathbb{Z}}^{gm}(\mathcal{M}_{0,n})$ and $\mathrm{DMT}_{/\mathrm{Spec}(\mathbb{Z}),\mathbb{Z}}^{gm}(\mathcal{M}_{0,n_1} \times \mathcal{M}_{0,n_2})$. Tangential base points lead to functors

$$\mathrm{DMT}_{/\mathrm{Spec}(\mathbb{Z}),\mathbb{Z}}^{gm}(\mathrm{Spec}(\mathbb{Z})) \longrightarrow \mathrm{DMT}_{/\mathrm{Spec}(\mathbb{Z}),\mathbb{Z}}^{gm}(\mathcal{M}_{0,n}).$$

Functoriality with respect to forgetful morphisms is a consequence of M. Spitzweck construction. Working over $\mathrm{Spec}(\mathbb{Z})$ with integral coefficients, these categories are equivalent to categories of perfect representations of affine derived group schemes. The above functorialities lead, as a conclusion of this section, to the definition of a motivic Grothendieck-Teichmüller group in this setting.

- Section 5 derives some consequences of the above constructions in more classical settings. In particular, working with \mathbb{Q} -coefficients, one obtains a tannakian group associated to the tannikian category given by the heart of the t -structure of $\mathrm{DMT}_{/\mathrm{Spec}(\mathbb{Z}),\mathbb{Q}}^{gm}(\mathcal{M}_{0,n})$. This leads to a motivic Grothendieck-Teichmüller group defined in terms of automorphisms of groups (an not derived groups). Working over $\mathrm{Spec}(\mathbb{F})$, the spectrum of a number field, we show how Deligne-Goncharov category of mixed Tate motives of the ring of its integers, agrees with M. Spitzweck construction of mixed motives and how our construction passes to this context. The end of the section presents the relation with M. Levine’s approach to mixed Tate motives and algebraic cycles.
- The last section is devoted to some conjectures about the “geometric” (derived) groups defining the motivic Grothendieck-Teichmüller group and its relations to Betti and De Rham realizations.

2. SHORT REVIEW OF SPITZWECK’S MIXED MOTIVES CATEGORY

Let S be noetherian separated scheme of finite Krull dimension. In [Spi13], Spitzweck built a E_∞ -ring object MZ_S in the category $\mathrm{Spt}_{\mathbb{P}^1}^\Sigma(S)$ of motivic symmetric \mathbb{P}^1 -spectra (cf. [Jar00, DLØ⁺07, Hov01]). The \mathbb{P}^1 -spectrum MZ_S serves in particular as the motivic Eilenberg-MacLane spectrum. It is also an oriented ring spectrum, that is in $\mathrm{SH}(S)$ it is an algebra over \mathbf{MGL} , the algebraic cobordism spectrum. Considering the category $\mathrm{Mod}_{\mathrm{MZ}_S}$ of modules over MZ_S , Spitzweck used a model structure on $\mathrm{Mod}_{\mathrm{MZ}_S}$ compatible with the one on $\mathrm{Spt}_{\mathbb{P}^1}^\Sigma(S)$ and defined a triangulated category $\mathrm{DM}_{\mathbb{Z}}(S)$ of motives over S with integral coefficients together with the following adjoint functor:

$$\begin{array}{ccc} \mathrm{Mod}_{\mathrm{MZ}_S} & \longrightarrow & \mathrm{Spt}_{\mathbb{P}^1}^\Sigma(S) \\ & \swarrow \scriptstyle \otimes \mathrm{MZ}_S & \end{array}$$

and

$$\begin{array}{ccc} \mathrm{DM}_{\mathbb{Z}}(S) & \longrightarrow & \mathrm{SH}(S) \\ & \swarrow \scriptstyle \otimes \mathrm{MZ}_S & \end{array}$$

where the left to right functors \longrightarrow are forgetful functors and the tensor products is the one given by symmetric monoidal structure of $\mathrm{Spt}_{\mathbb{P}^1}^{\Sigma}(S)$ (corresponding to the smash product \wedge in [Jar00]).

We recall below some definitions and properties needed for our construction of motivic Grothendieck-Teichmüller group. Our construction is geometric and is based on the most expected distinguished triangles in $\mathrm{DM}_{\mathbb{Z}}(S)$ and on the functoriality of its construction. In particular, we recall below Gysin's distinguished triangles and Blow-ups formula in Spitzweck's category. Because of the existence of Chern class in Spitzweck category and its relation with the stable motivic homotopy category, these are direct consequences of Déglise works [Dég08]. Working over a number field and with \mathbb{Q} coefficient would cancel the need of the following subsections as distinguished triangles where proved in [Voe00] and functoriality for the associated mixed Tate categories would be insured by Levine's work [Lev10].

2.1. Symmetric spectrum, $\mathrm{SH}(X)$ and mixed motives. Let Sm_S denote the category of smooth schemes of finite type over S and $\mathrm{Sm}_S|_{\mathrm{Nis}}$ the smooth Nisnevich site over S . We recall below some facts about Spitzweck construction [Spi13] and we are mostly interested in the case where S is $\mathrm{Spec}(\mathbb{Z})$.

Let $\mathrm{Spc}(S)$ be the category of (motivic) spaces over S , that is of Nisnevich sheaves over S with value into simplicial sets. M. Spitzweck construction actually uses complexes of sheaves of abelian groups. Classical comparison functor and transfer of structures insure that his construction passes to motivic spectra. Via the Yoneda embedding any scheme in Sm_S is a motivic space (constant in the simplicial direction); any simplicial set is also a motivic space as a constant sheaf. The terminal object is represented by S itself.

A pointed (motivic) space is a motivic space X together with a map

$$x : S \longrightarrow X.$$

The category of pointed spaces is denoted by $\mathrm{Spc}_{\bullet}(S)$. To any space X , one associates a canonical pointed space $X_+ = X \sqcup *$. The category $\mathrm{Spc}_{\bullet}(S)$ admits a monoidal structure \otimes induced by the smash product on pointed simplicial sets.

Recall that the simplicial circle is the coequalizer of

$$\Delta[0] \rightrightarrows \Delta[1]$$

and let S_s^1 be the corresponding pointed space. Moreover, let S_T^1 , the Tate circle, be the pointed space represented by $(\mathbb{P}^1, \{\infty\})$.

Very shortly, a symmetric \mathbb{P}^1 -spectrum E is a collection of pointed spaces $E = (E_0, E_1, \dots)$ with structure maps $S_T^1 \otimes E_n \rightarrow E_{n+1}$ and with the extra data of a symmetric group actions $\Sigma_n \times E_n \rightarrow E_n$ such that the composition maps

$$(S_T^1)^{\otimes p} \otimes E_n \longrightarrow E_{n+p}$$

are $\Sigma_p \times \Sigma_n$ equivariant.

The iterated products of S_s^1 (resp. S_T^1) are denoted by S_s^n (resp. S_T^m). Tensoring with the simplicial circle (resp. the Tate circle) induce a simplicial (resp. a Tate) suspension functor denoted by Σ_s^1 (resp. Σ_T^1). Any motivic space X induces a symmetric \mathbb{P}^1 -spectrum

$$\Sigma_T^{\infty} X_+ = (X_+, S_T^1 \otimes X_+, S_T^2 \otimes X_+, \dots).$$

We denote by $\mathrm{Spt}_{\mathbb{P}^1}^{\Sigma}(S)$ the category of symmetric \mathbb{P}^1 -spectra. The *(motivic) stable homotopy category* $\mathrm{SH}(S)$ is obtained from $\mathrm{Spt}_{\mathbb{P}^1}^{\Sigma}(S)$ by inverting stable weak equivalence [Hov01]. In particular, the suspension functors Σ_s and Σ_T are invertible as are \mathbb{A}^1 weak equivalences.

The category $\mathrm{SH}(S)$ is a triangulated category with shift induced by Σ_s^1 . In $\mathrm{SH}(S)$ the suspension functor Σ_s will be denoted by the shift notation $[1]$. Note that S_T^1 is isomorphic to $S_s^1 \otimes (\mathbb{G}_m, \{1\})$ in $\mathrm{SH}(S)$.

Spitzweck, in [Spi13, Definition 4.27], defined a \mathbb{P}^1 -spectrum MZ_S or simply MZ when S is clear enough. The \mathbb{P}^1 -spectrum MZ_S is an E_∞ -ring object in $\mathrm{Spt}_{\mathbb{P}^1}^\Sigma(S)$ and induces a ring object in $\mathrm{SH}(S)$ again denoted by MZ_S .

The category of motives $\mathrm{DM}_{\mathbb{Z}}(S)$ is defined as the homotopy category of modules (in \mathbb{P}^1 -spectra) over MZ_S . For any $X \in \mathrm{Sm}_S$, the category $\mathrm{DM}_{\mathbb{Z}}(X)$ is defined similarly as the homotopy category of modules over $f^*\mathrm{MZ}_S$ where $f : X \rightarrow S$ is the structural morphism and $f^* : \mathrm{Spt}_{\mathbb{P}^1}^\Sigma(S) \rightarrow \mathrm{Spt}_{\mathbb{P}^1}^\Sigma(X)$ the pull-back functor between spectra categories. In the case when we need to remember over which base S we are working, we may write $\mathrm{DM}_{/S, \mathbb{Z}}(X)$.

For $X \xrightarrow{f} S$ in Sm_S , we have a functor

$$\mathrm{Sm}_X \xrightarrow{M_X} \mathrm{DM}_{\mathbb{Z}}(X)$$

$$Y \longmapsto M_X(Y) = \Sigma_{\mathbb{P}^1}^\infty(Y_+) \otimes f^*\mathrm{MZ}_S$$

In $\mathrm{DM}_{\mathbb{Z}}(X)$, the tensor unit $f^*\mathrm{MZ}_S$ will be denoted by $\mathbb{Z}_X(0)$. The Tate object $\mathbb{Z}_X(1)$ is defined by

$$\mathbb{Z}_X(1)[2] = \Sigma_T^1 \mathbb{Z}_X(0) = (\mathbb{P}^1, \infty) \otimes f^*\mathrm{MZ}_S$$

and corresponds as usually to the cone of the morphism

$$M_X(\{\infty\}) \longrightarrow M_X(\mathbb{P}^1)$$

shifted by -2 .

The suspension $\Sigma_s^{n-2p} \circ \Sigma_T^p$ will be denoted by $\Sigma^{n,p}$.

Remark 2.1. M. Spitzweck shows in [Spi13, Section 10] that the functor $X \rightarrow \mathrm{DM}_{\mathbb{Z}}(X)$ satisfies the 6 functors formalism.

2.2. An oriented cohomology theory. For $f : X \rightarrow S$ a smooth scheme over S , Spitzweck showed [Spi13, Proposition 11.1] that the ring objects MZ_S and $f^*\mathrm{MZ}_S$ are oriented in the sens of F. Morel and G. Vezzosi [Vez01], that is there is a distinguished element

$$\nu \in \mathrm{Hom}_{\mathrm{SH}(X)}(\Sigma^\infty(\mathbb{P}^\infty_+), \Sigma^{2,1} f^*\mathrm{MZ}_S),$$

where \mathbb{P}^∞ denotes the colimit of the \mathbb{P}^n , such that ν restricts to the canonical element induced by the unit of $f^*\mathrm{MZ}_S$ in $\mathrm{Hom}_{\mathrm{SH}(S)}(\Sigma^\infty(\mathbb{P}^1_+), \Sigma^{2,1} f^*\mathrm{MZ}_S)$.

When S is regular, the morphism

$$\mathrm{Pic}(Y) \longrightarrow \mathrm{Hom}_{\mathrm{SH}(S)}(\Sigma^\infty(Y_+), \Sigma^\infty(\mathbb{P}^\infty_+)),$$

for any $Y \rightarrow X$ smooth, is an isomorphism and endows $\mathrm{DM}_{\mathbb{Z}}(X)$ with an orientation as described by F. Déglise in [Dég08, 2.1-(Orient) axiom] (see [Dég08, 2.3.2] or [MV99, Proposition 4.3.8]). That is, for any $Y \rightarrow X$ smooth there is an application, called *first Chern class*:

$$c_1 : \mathrm{Pic}(Y) \longrightarrow \mathrm{Hom}_{\mathrm{DM}_{\mathbb{Z}}(X)}(M_X(Y), \mathbb{Z}_X(1)[2])$$

which is functorial in Y and such that the image of the canonical bundle on \mathbb{P}_X^1 is the canonical projection.

Note that the formal group law attached to the first Chern class is the additive ones ([Spi13, Theorem 7.10])

Thanks to F. Déglise's work in [Dég08], one obtains then the following properties:

2.3. Distinguished triangles and split formulas in $\mathrm{DM}_{\mathbb{Z}}(X)$. Let Y be a smooth scheme in Sm_X and $p : P \rightarrow Y$ a projective bundle over Y of rank n . We denote by λ the canonical line bundle over P and put

$$c = c_1(\lambda) : M_X(P) \rightarrow \mathbb{Z}_X(1)[2].$$

The diagonal $\delta_i : P \rightarrow \underbrace{P \times_Y \cdots \times_Y P}_{i+1 \text{ times}}$ followed by $p_* \otimes c^{\otimes i}$ gives a morphism

$$\epsilon_{P,i} : M_X(P) \rightarrow M_X(Y)(i)[2i].$$

Proposition 2.2 (Projective bundle formula ([Dég08, Theorem 3.2])). *With the above notation, the morphism*

$$\epsilon_P : M_X(P) \rightarrow \bigoplus_{i=1}^n M_X(Y)(i)[2i] \quad \epsilon = \sum_{i=0}^n \epsilon_{P,i}$$

is an isomorphism.

For any $0 \leq r \leq n$, one can now define the embedding

$$\iota_i : M_X(Y)(r)[2r] \xrightarrow{*(-1)^r} \bigoplus_{i=1}^n M_X(Y)(i)[2i] \rightarrow M_X(P).$$

Let Z be a smooth closed subscheme of Y smooth such that Z is everywhere of codimension n . As in more general situations, one defines the motive of Y with support in Z as

$$M_{X,\mathrm{Supp}(Z)}(Y) = M(Y/(Y \setminus Z)).$$

Then F. Déglise in Proposition 4.3 of [Dég08], attached to the pair (Z, Y) a unique isomorphism

$$(\text{purity}) \quad \mathfrak{p}_{Y,Z} : M_{X,\mathrm{Supp}(Z)}(Y) \rightarrow M_X(Z)(n)[2n]$$

which is functorial with respect to Cartesian morphism of such pairs and such that, when E is a vector bundle over Y of rank n and $P = \mathbb{P}(E \oplus 1)$, $\mathfrak{p}_{P,X}$ is the inverse of

$$M_X(Y)(n)[2n] \xrightarrow{\iota_n} M_X(P) \rightarrow M_{X,\mathrm{Supp}(Y)}(P).$$

Defining the Thom motive $M_X \mathrm{Th}_Y(E)$ of a rank n vector bundle E over Y , one obtains the Thom isomorphism [Dég08, §4.4]:

$$\mathfrak{p}_{E,Y} : M_X \mathrm{Th}_Y(E) \rightarrow M_X(n)[2n].$$

The purity isomorphism allows us to rewrite the localization distinguished triangle as follows.

Proposition 2.3 (Gysin triangle [Dég08, Definition 4.6]). *Let Z be smooth closed subscheme of Y smooth such that Z is everywhere of codimension n . Then there is a distinguished triangle*

$$(1) \quad M_X(Y \setminus Z) \xrightarrow{j_*} M_X(Y) \xrightarrow{i^*} M_X(Z)(n)[2n] \xrightarrow{\partial_{Y,Z}} M_X(Y \setminus Z)[1]$$

where i^* (resp. $\partial_{X,Z}$) is called the Gysin morphism (resp. residue morphism).

The Gysin triangle is functorial and in particular compatible with the projective bundle isomorphisms and with the induced embeddings ι_r . Moreover Gysin morphisms are multiplicative with respect to compositions and products [Dég08, corollaries 4.33 and 4.34].

Using the projective bundle formula first in the case of $Y = X$, F. Déglise showed that $p : \mathbb{P}_X^n \rightarrow X$ admits a strong dual given by [Dég08, Definition 5.6]:

$$M_X(\mathbb{P}_X^n)(-n)[-2n]$$

together with a Gysin morphism $p^* : M_X(X) \longrightarrow M_X(\mathbb{P}_X^n)(-n)[-2n]$. This duality allows to define a Gysin morphism $p_*^* : M_X(X) \longrightarrow M_X(\mathbb{P}_X^n)(-n)[-2]$ as the transpose of p_* . Then, generalizing this situation to smooth projective X -schemes and factorizing a projective morphism $f : Y_1 \longrightarrow Y_2$ between such as a closed immersion and a projection $f = p \circ i$, F. Déglise obtained a Gysin morphism [Dég08, Definition 5.12]

$$f^* := i^* \circ p^* : M_X(Y_2) \longrightarrow M_X(Y_1)(-d)[-2d]$$

where i is of codimension $d + n$ into $\mathbb{P}_{Y_2}^n$

Now, let $p : Y \longrightarrow X$ be a smooth projective scheme over X of pure dimension n . The above Gysin morphisms lead to the definition of

$$\mu_Y : \mathbb{Z}_X(0) \xrightarrow{p^*} M_X(Y)(-n)[-2n] \xrightarrow{\delta^*} M_X(Y)(-n)[-2n] \otimes M_X(Y)$$

and

$$\epsilon_Y : M_X(Y) \otimes M_X(Y)(-n)[-2n] \xrightarrow{\delta^*} M_X(Y) \xrightarrow{p_*} \mathbb{Z}_X(0)$$

where δ is the diagonal $\delta : Y \longrightarrow Y \times_X Y$.

Theorem 2.4 (Duality - [Dég08, Theorem 5.23 and Proposition 5.26]). *Let $p : Y \longrightarrow X$ be a smooth projective scheme over X . The motive $M_X(Y)(-n)[-2n]$, endowed with μ_Y and ϵ_Y , is a strong dual of $M(X)$.*

Proposition 2.5 (Blow-up formula - [Dég08, Theorem 5.38]). *Let Y be a smooth scheme over X and Z a smooth closed subscheme of Y purely of codimension n .*

Let $B_Z(Y)$ be the blow-up of Y with center Z and E_Z be the exceptional divisor, then:

$$M(B_Z(Y)) \simeq M_X(Y) \oplus \bigoplus_{i=1}^{n-1} M_X(Z)(i)[2i].$$

2.4. Beilinson-Soulé's Vanishing property. M. Levine, in [Lev10], proved that if X is a smooth variety over a number field \mathbb{F} with a motive of mixed Tate type and satisfying Beilinson-Soulé vanishing property (see (BS)) then there exists a well defined tannakian (in particular \mathbb{Q} linear) category $\text{MTM}_{/\mathbb{F}, \mathbb{Q}}(X)$ of mixed Tate motives over X ([Lev10, Theorem 3.6.9]) together with a short exact sequence relating the tannakian groups of $\text{MTM}_{/\mathbb{F}, \mathbb{Q}}(\mathbb{F})$ and of $\text{MTM}_{/\mathbb{F}, \mathbb{Q}}(X)$ ([Lev10, Section 6.6]). This short exact sequence is a motivic avatar of the short exact sequence for étale fundamental groups relating $\text{Gal}(\overline{\mathbb{F}}/\mathbb{F})$ and $\pi_1^{\text{ét}}(X)$.

In a similar direction, M. Spitzweck obtains in [Spi10], that the triangulated category $\text{DMT}_{/\mathbb{F}, \mathbb{Z}}^{\text{gm}}(X)$ of mixed Tate motives over X (that is before applying a t -structure and obtaining $\text{MTM}(-)$) is the category $\text{Perf}(B_X^\bullet)$ of perfect representations of an affine derived group scheme over \mathbb{Z} provided that X is a smooth connected \mathbb{F} -scheme of finite type (\mathbb{F} is any field) satisfying a weaker Beilinson-Soulé's vanishing property (wBS) (see [Spi10, Theorem 2.2]). Corollary 8.4 in [Spi13] extends this construction to the case where X is simply smooth over S and satisfying (wBS). We give the necessary definition and results below.

M. Spitzweck showed at corollaries 7.19 and 7.20 in [Spi13] that his construction retrieves motivic cohomology.

Proposition 2.6 ([Spi13, 7.19 and 7.20]). *For a smooth scheme X over $S = \text{Spec}(D)$ the spectrum of a Dedekind domain of mixed characteristic, one has*

$$\begin{aligned} \text{Hom}_{\text{SH}(S)}(\Sigma^\infty(X_+), \text{MZ}_S(p)[k]) &\simeq \text{Hom}_{\text{DM}(S)}(M_S(X), \mathbb{Z}_S(p)[k]) \\ &\simeq \text{Hom}_{\text{DM}(X)}(\mathbb{Z}_X(0), \mathbb{Z}_X(p)[k]) \\ &\simeq H_{\text{mot}}^k(X, p) \end{aligned}$$

where $H_{mot}^k(X, p)$ denotes the motivic cohomology in the sense of Levine [Lev98]; that is it gives back the higher Chow groups (see [Blo86, BK94, Lev94, Lev10]) of X :

$$(2) \quad H_{mot}^k(X, p) = CH^p(X, 2p - k)$$

Definition 2.7 (Beilinson-Soulé vanishing property). Let X be a smooth scheme over S . One says that X satisfies Beilinson-Soulé vanishing property (BS) if and only if

$$(BS) \quad \text{Hom}_{SH(S)}(\Sigma^\infty(X_+), M\mathbb{Z}_S(p)[k]) = 0,$$

for all $p \geq 0$ and $k < 0$ and for all $p > 0$ and $k = 0$.

M. Spitzweck often needs only a weaker form of this property.

Definition 2.8 (weak Beilinson-Soulé vanishing property). Let X be a smooth scheme over S . One says that X satisfies Beilinson-Soulé vanishing property (wBS) if and only if

$$(wBS) \quad \forall p > 0, \quad \exists N \in \mathbb{Z}, \text{ such that } \text{and } k < N,$$

$$\text{Hom}_{SH(S)}(\Sigma^\infty(X_+), M\mathbb{Z}_S(p)[k]) = 0.$$

Remark 2.9. Note also as a consequence of [Spi13, in particular Theorem 7.10], one has also for X a smooth irreducible S -scheme (S regular)

$$\text{Hom}_{DM(S)}(M_S(X), \mathbb{Z}(p)[k]) = \begin{cases} 0 & \text{for } p < 0 \\ 0 & \text{for } p = 0 \text{ and } k \neq 0 \\ \mathbb{Z} & \text{for } p = 0 \text{ and } k = 0 \\ \mathcal{O}_X(X)^* & \text{for } p = 1 \text{ and } k = 1 \\ \text{Pic}(X) & \text{for } p = 1 \text{ and } k = 2 \end{cases}$$

Theorem 2.10. $S = \text{Spec}(\mathbb{Z})$ satisfies (BS) property.

Proof. The work of Borel [Bor74] and Beilinson [Beĭ84] and the comparison between groups of K -theory and motivic cohomology through higher Chow groups, shows that with \mathbb{Q} coefficients $\text{Spec}(\mathbb{Q})$ satisfies the (BS) property. The difference between $\text{Spec}(\mathbb{Z})$ and $\text{Spec} \mathbb{Q}$ is concentrated in degree 1 weight 1 where the latter has an extra generator for each prime. Thus $\text{Spec}(\mathbb{Z})$ satisfies the (BS) property with \mathbb{Q} coefficient. As reviewed in [Kah05, lemma 24], the (BS) property with \mathbb{Z} -coefficient is a consequence of the Beilinson-Soulé vanishing property with \mathbb{Q} coefficients together with the Beilinson-Lichtenbaum conjecture [Kah05, Conjecture 17] which is equivalent to the Bloch-Kato conjecture (see. [SV00]). Thanks to the work of V. Voevodsky this last conjecture is a now a theorem proved in [Voe11]. This concludes our theorem. \square

Remark 2.11. Let \mathbb{F} be a number field and \mathcal{P} a set of finite place of \mathbb{F} . Note that similar arguments show that the Beilinson-Soulé vanishing property also holds when S is the spectrum of the ring $\mathcal{O}_{\mathbb{F}, \mathcal{P}}$ of \mathcal{P} -integers of \mathbb{F} ; that is when

$$S = \text{Spec}(\mathcal{O}_{\mathbb{F}, \mathcal{P}}).$$

3. GEOMETRY OF MODULI SPACES $\overline{\mathcal{M}}_{0,n}$

Let n be an integer greater or equal to three and $\mathcal{M}_{0,n}$ be the moduli space of curves of genus 0 with n marked points over $\text{Spec}(\mathbb{Z})$ and $\overline{\mathcal{M}}_{0,n}$ its Deligne-Mumford compactification [DM69, Knu83]. The integer $l = n - 3$ is the dimension of $\mathcal{M}_{0,n}$ and the boundary $\partial \overline{\mathcal{M}}_{0,n} = \overline{\mathcal{M}}_{0,n} \setminus \mathcal{M}_{0,n}$ is a strict normal crossing divisor whose irreducible components are isomorphic to $\overline{\mathcal{M}}_{0,n_1} \times \overline{\mathcal{M}}_{0,n_2}$ with $n_1 + n_2 = n + 2$. For \mathbb{F} a number field, we shall write $\mathcal{M}_{0,n/\mathbb{F}}$ and $\overline{\mathcal{M}}_{0,n/\mathbb{F}}$ when the moduli spaces are

considered over $\text{Spec } \mathbb{F}$ and when the context needs to be precised. If S is a finite set with n elements, we will write $\mathcal{M}_{0,S}$ and $\overline{\mathcal{M}}_{0,S}$ for $\mathcal{M}_{0,n}$ and $\overline{\mathcal{M}}_{0,n}$. Note that if $(\mathbb{P}^1)_*^{|S|}$ denotes the set of all n -tuples of distinct points $z_s \in \mathbb{P}^1$, for $s \in S$, then

$$\mathcal{M}_{0,S} = \text{PSL}_2 \backslash (\mathbb{P}^1)_*^{|S|},$$

where PSL_2 is the algebraic group of automorphisms of \mathbb{P}^1 and acts by Möbius transformations.

Let $S = \{1, \dots, n\}$. Recall that for any subset S' of S , there exists a natural map

$$f_{S'} : \mathcal{M}_{0,S} \longrightarrow \mathcal{M}_{0,S'}$$

obtained by forgetting the marked points of S which do not lie in S' . This maps extends to a proper morphism

$$(3) \quad \overline{f_{S'}} = \overline{\mathcal{M}_{0,S}} \longrightarrow \overline{\mathcal{M}_{0,S'}}.$$

3.1. On the boundary of $\overline{\mathcal{M}}_{0,n}$. Let D be a codimension 1 irreducible component of $\partial \overline{\mathcal{M}}_{0,n}$ and D_0 its open strata:

$$D_0 = D \setminus \left(\bigcup_{D' \neq D} D \cap D' \right)$$

where the unions runs through the codimension 1 irreducible components of $\partial \overline{\mathcal{M}}_{0,n}$ different from D . The union

$$\mathcal{M}_{0,n} \cup D_0 = \overline{\mathcal{M}}_{0,n} \setminus (\partial \overline{\mathcal{M}}_{0,n} \setminus D_0)$$

is denoted by $\mathcal{M}_{0,n}^D$ and the normal bundle of D_0 in $\mathcal{M}_{0,n}^D$ is denoted by N_{D_0} . The goal of this section is to prove that N_{D_0} is trivial.

Let S denote the set $\{1, \dots, n\}$. The moduli space $\overline{\mathcal{M}}_{0,S}$ admits a stratification (cf. [1]). The open strata is simply $\mathcal{M}_{0,S}$. A point in a codimension k strata represents a stable curve with n marked points which is a tree of \mathbb{P}^1 's (because of the genus 0) with the n marked points spread on the branches such that each \mathbb{P}^1 , that is each branch, has at least three special points (marked points and intersection points). Moving into a strata makes the marked points move into their branch but they can not move from one branch to another. Following Keel's work [Kee92], an irreducible component D of $\partial \mathcal{M}_{0,S}$ is given by its open strata. A point in this open strata represents a tree of \mathbb{P}^1 with two branches, the marked points being spread on these two branches. Thus D gives a two partition of S which determines the open stratum of D . D is then given by a subset T_D of S such that T_D and its complements T_D^C have at least 2 elements and D is isomorphic to $\overline{\mathcal{M}}_{0,T \cup \{e\}} \times \overline{\mathcal{M}}_{0,T^C \cup \{e\}}$ with e not in S .

We fix three elements i_0, i_1, i_2 in S . The correspondence between codimension 1 irreducible components of $\partial \overline{\mathcal{M}}_{0,n}$ and partition $J \sqcup J^c$ of S is now made 1-to-1 by imposing that $|J \cap \{i_0, i_1, i_2\}| \leq 1$. For such a J , the corresponding component of $\partial \overline{\mathcal{M}}_{0,n}$ is denoted by D^J .

We shall use the following notations. When the ambient space should be pointed out, we will write D_S^J instead of D^J . The open stratum of D^J is

$$D_{0,S}^J = D_S^J \setminus (\cup_{D' \neq D_S^J} D_S^J \cap D')$$

where the union runs through the codimension 1 irreducible components D' of $\partial \text{mob}[S]$ different from D_S^J . Following the above notation, the union $\mathcal{M}_{0,S} \cup D_{0,S}^J$ is denoted by $\overline{\mathcal{M}}_{0,S}^{D^J}$. Note that

$$\overline{\mathcal{M}}_{0,S}^{D^J} = \overline{\mathcal{M}}_{0,S} \setminus (\cup_{D' \neq D_S^J} D_S^J \cap D')$$

where the union runs as above.

We suppose that $n \geq 5$, thus we can assume that T_D^c has at least 3 elements. Let $I = \{i_0, i_1, i_2, i_3\}$ be a subset of S such that i_0 is in T and i_1, i_2 and i_3 are elements of T^c . Let S_0 be $S \setminus \{i_3\}$. We consider the morphism

$$\pi_{S_0 \times I} : \overline{\mathcal{M}_{0,S}} \xrightarrow{\overline{f_{S_0} \times f_I}} \overline{\mathcal{M}_{0,S_0}} \times \overline{\mathcal{M}_{0,I}}.$$

Lemma 3.1. *Let $S = \{1, \dots, n\}$, $T \subset S$, $I = i_0, i_1, i_2, i_3$ and S_0 be as above. Then the image of D_0^T by $\pi_{S_0 \times I}$ satisfies:*

$$\pi_{S_0 \times I}(D_0^T) \subset D_{0,S_0}^T \times \mathcal{M}_{0,4}.$$

Proof. Let P be a point in D_0^T . As P is in the open strata, P represents a tree of \mathbb{P}^1 having only two branches with n marked points spread on each branch accordingly to the partition $T \sqcup T^c$ of S . The forgetful morphisms make at worst the number of branches decrease. Hence $\overline{f_{S_0}}(P)$ have at most 2 branches and thus is at worst in the open strata of a codimension 1 irreducible component of $\overline{\mathcal{M}_{0,S_0}}$. In one hand, note that the as $|T| \geq 2$ and $i_3 \notin T$, $\overline{f_{S_0}}(P)$ is in $D_{S_0}^T$ and thus is in D_{0,S_0}^T . In the other hand $T \cap \{i_0, i_1, i_2, i_3\} = \{i_0\}$ and the tree of \mathbb{P}^1 corresponding to P can not remain stable under $\overline{f_I}$, thus $\overline{f_I}(P)$ represents a single \mathbb{P}^1 with n marked points and thus is in $\mathcal{M}_{0,4}$. \square

Proposition 3.2. *Let n be greater or equal to 4 and let D be a codimension 1 irreducible component of $\partial \overline{\mathcal{M}_{0,n}}$. Then, the normal bundle N_{D_0} is trivial.*

Proof. The proof proceed by induction on n and is clear for $n = 4$.

Assuming that $n \geq 5$, we write as above $S = \{1, \dots, nn\}$ and D as D^T for some $T \subset S$. The cardinal of S being at least 5, we can assume that $|T| \geq 2$ and that $|T^c| \geq 3$. In order to have a 1-to-1 correspondence between codimension 1 irreducible component of $\partial \overline{\mathcal{M}_{0,S}}$ and S partition as described above, we chose i_0 in T and i_1 and i_2 in T^c . T^c having at least three elements, we chose there a third elements i_3 different from i_1 and i_2 .

Keel's work [Kee92, Lemma 1] shows that the morphism

$$\pi_{S_0 \times I} : \overline{\mathcal{M}_{0,S}} \xrightarrow{\overline{f_{S_0} \times f_I}} \overline{\mathcal{M}_{0,S_0}} \times \overline{\mathcal{M}_{0,I}}$$

is given by a succession of blow-ups along regular, smooth, codimension 2 subschemes and whose exceptional divisors are codimension 1 irreducible components of $\partial \overline{\mathcal{M}_{0,n}}$ of the form :

$$D^{J \cup \{3\}} \quad \text{with} \quad \begin{cases} J \subset S_0, & |J| \geq 2 \\ \text{and } |J \cap \{i_0, i_1, i_2\}| \leq 1. \end{cases}$$

In particular, $\pi_{S_0 \times I}$ is an isomorphism outside the exceptional divisor. Hence, the image of $\mathcal{M}_{0,S}^{D^T}$ by $\pi_{S_0 \times I}$ is an open of $\overline{\mathcal{M}_{0,S_0}} \times \overline{\mathcal{M}_{0,4}}$.

As T is also a subset of S_0 , let $D_{S_0}^T$ be the corresponding codimension 1 component of $\partial \overline{\mathcal{M}_{0,S_0}}$ and D_{0,S_0}^T its open stratum. Lemma 3.1 above shows that

$$\pi_{S_0 \times I}(D_0^T) \subset D_{0,S_0}^T \times \mathcal{M}_{0,4}.$$

Thus, the image of $\pi_{S_0 \times I}(\mathcal{M}_{0,S}^{D^T})$ is open in $\overline{\mathcal{M}_{0,S_0}} \times \overline{\mathcal{M}_{0,4}}$ and is included in $\mathcal{M}_{0,S_0}^{D^T} \times \overline{\mathcal{M}_{0,4}}$ which is also open in $\overline{\mathcal{M}_{0,S_0}} \times \overline{\mathcal{M}_{0,4}}$. As a consequence, $\pi_{S_0 \times I}(\mathcal{M}_{0,S}^{D^T})$ is open in $\mathcal{M}_{0,S_0}^{D^T} \times \overline{\mathcal{M}_{0,4}}$.

The morphism $\pi_{S_0 \times I}$ being an isomorphism away from the exceptional divisor, the triviality of $N_{D_0^T}$ in $\mathcal{M}_{0,S}^{D^T}$ is equivalent to the triviality of the normal bundle of $\pi_{S_0 \times I}(D_0^T)$ in $\pi_{S_0 \times I}(\mathcal{M}_{0,S}^{D^T})$ which by the above discussion is a consequence of

the triviality of N_{D_{0,S_0}^T} in $\mathcal{M}_{0,S_0}^{D_{0,S_0}^T}$. The Proposition follows by induction, the case $\overline{\mathcal{M}_{0,4}} \simeq \mathbb{P}^1$ with $\partial\overline{\mathcal{M}_{0,4}} \simeq \{0, 1, \infty\}$ being trivial. \square

3.2. The motives of $\overline{\mathcal{M}_{0,n}}$. Now, $S = \text{Spec}(\mathbb{Z})$. The main goals of this subsection is to prove that the motive $M_S(\overline{\mathcal{M}_{0,n}})$ is a (finite) direct sum of motives of the type $\mathbb{Z}_S(p)[2p]$ with $p \geq 0$ and satisfy (BS) property (see Theorem 3.5).

The key points are the decomposition of $M_S(\overline{\mathcal{M}_{0,n}})$ into Tate motives and the Beilinson-Soulé property for the base scheme $S = \text{Spec}(\mathbb{Z})$.

Definition 3.3. Let X be a smooth scheme over S . We say that X is *effective of type Tate type* $(p, 2p)$ or simply of type *ET* when $M_S(X)$ is a finite direct sum of motives $\mathbb{Z}_S(p_i)[2p_i]$ with $p \geq 0$.

A direct application of the blow-up formula (Proposition 2.5) gives the following.

Lemma 3.4. *Let X be a smooth scheme over S and Z a smooth closed subscheme of X . We assume that both X and Z are of type ET. Then the blow-up $\text{Bl}_Z(X)$ of X with center Z is also of type ET.*

Theorem 3.5. *Let n be an integer greater or equal to 3. The motive $M_S(\overline{\mathcal{M}_{0,n}})$ is isomorphic to*

$$M_S(\overline{\mathcal{M}_{0,n}}) = \oplus_i \mathbb{Z}_S(p_i)[2p_i]$$

where the above sum is finite and the p_i are positive (or zero). Moreover $M_S(\overline{\mathcal{M}_{0,n}})$ satisfy (BS); that is for any integers p and k such that $p \geq 0$ and $k < 0$, or $p > 0$ and $k = 0$ one has

$$\text{Hom}_{\text{SH}(S)}(\Sigma^\infty(\overline{\mathcal{M}_{0,n_+}}), \mathbb{M}\mathbb{Z}_S(p)[k]) = \text{Hom}_{\text{DM}(S)}(M_S(\overline{\mathcal{M}_{0,n}}), \mathbb{Z}_S(p)[k]) = 0$$

Proof. Note that the second part of the theorem follows directly from the first part using Lemma 3.6 below.

The proof of the first part is by induction on n .

Note that $\overline{\mathcal{M}_{0,3}}$ is isomorphic to $S = \text{Spec}(\mathbb{Z})$ and $\overline{\mathcal{M}_{0,4}}$ is simply \mathbb{P}_S^1 . Hence, using the (BS) property for $S = \text{Spec}(\mathbb{Z})$ (Lemma 2.10) and the projective bundle formula, $\overline{\mathcal{M}_{0,3}}$ and $\overline{\mathcal{M}_{0,4}}$ are of type ET.

Fix $n \geq 5$. Let I_n denotes the set $\{1, \dots, n\}$ (denoted by S in the previous section). Keel in [Kee92, Theorem 1 and 2] proves that the morphism

$$\overline{\mathcal{M}_{0,I_n}} \longrightarrow \overline{\mathcal{M}_{0,I_{n-1}}} \times \overline{\mathcal{M}_{0,I_4}}$$

is a sequence of blow-ups

$$\overline{\mathcal{M}_{0,I_n}} \xrightarrow{\sim} B_{n-3} \rightarrow \cdots \rightarrow B_k \rightarrow \cdots \rightarrow B_1 = \overline{\mathcal{M}_{0,I_{n-1}}} \times \overline{\mathcal{M}_{0,I_4}}$$

where $B_{k+1} \rightarrow B_k$ is the blow-up along disjoint centers isomorphic to some irreducible components of $\partial\overline{\mathcal{M}_{0,I_{n-1}}}$.

As $B_1 \simeq \overline{\mathcal{M}_{0,I_{n-1}}} \times \overline{\mathcal{M}_{0,I_4}}$, the induction hypothesis and Künneth formula show that B_1 is of type ET. Note that irreducible components of $\overline{\mathcal{M}_{0,I_{n-1}}}$ are isomorphic to $\overline{\mathcal{M}_{0,n_1}} \times \overline{\mathcal{M}_{0,n_2}}$ with $n_1 + n_2 = n - 1$. Thus Künneth formula and the induction hypothesis show that the centers of the blow-up $B_{k+1} \rightarrow B_k$ are also of type ET.

Now, using a proof similar to the proof of Proposition 4.4 in [Sou10], an induction on k and the blow-up formula prove that B_k is of type ET for all k . Hence $\overline{\mathcal{M}_{0,I_n}} \simeq B_{n-3}$ is also of type ET. \square

Note that [Sou10] use a cohomological setting which explains the minus signs in shift and twist. One could by-pass part of Keel's result in [Kee92] by remarking

that the map

$$\overline{\mathcal{M}_{0,I_n}} \longrightarrow \prod_{i=4}^n \overline{\mathcal{M}_{0,\{1,2,3,i\}}}$$

crashes down all irreducible components of the form D^T with $|T \cap \{1, 2, 3\}| \leq 1$ and $|T| \geq 3$. Normalizing the marked point z_1, z_2 and z_3 to $1, \infty$ and 0 respectively, one obtains that $\overline{\mathcal{M}_{0,I_n}}$ is the results of blowing up $(\mathbb{P}^1)^{n-3}$ along the poset given by the all the intersections of the divisors $t_i = t_j$ and $t_i = \varepsilon$ with $i \neq j$ and $\varepsilon = 0, 1, \infty$. This is exactly the situation of [Sou10, Proposition 4.4] and its proof has to be modified each times it uses the blow-up formula in order to take into account the ET type property.

Lemma 3.6. *Let X be a smooth scheme over S . Assume that X is effective of type Tate $(p, 2p)$, then $M_S(X)$ satisfy the Beilinson-Soulé property (BS).*

Proof. By definition, $M_S(X)$ is a direct sum of Tate motives of the form $\mathbb{Z}_S(i)[2i]$ for $i \geq 0$. Using Proposition 2.6, in order to show that $M_S(X)$ satisfies (BS), it then is enough to show that, for any $i \geq 0$ and for any $p \geq 0$ and any $k < 0$ or any $p > 0$ and $k = 0$, one has

$$\mathrm{Hom}_{\mathrm{DM}(S)}(\mathbb{Z}_S(i)[2i], \mathbb{Z}(p)[k]) = 0.$$

However the above Hom group is simply

$$\mathrm{Hom}_{\mathrm{DM}(S)}(\mathbb{Z}_S(0), \mathbb{Z}(p-i)[k-2i]).$$

If $p-i$ is less or equal to 0, one can use Remark 2.9. When $p-i > 0$, then the results follows from the (BS) property of $S = \mathrm{Spec}(\mathbb{Z})$ given at Theorem 2.10 because $k-2i < 0$. \square

Corollary 3.7. *Let $n \leq 4$ and D an irreducible component of $\partial\overline{\mathcal{M}_{0,n}}$. Then, D is of type ET and satisfies the (BS) property.*

Moreover, if \mathcal{S} is a non empty intersection of k irreducible codimension 1 components of $\partial\overline{\mathcal{M}_{0,n}}$, then \mathcal{S} is of type ET and satisfies the (BS) property.

Proof. The closed strata \mathcal{S} is isomorphic to a product

$$\overline{\mathcal{M}_{0,l_1+3}} \times \overline{\mathcal{M}_{0,l_2+3}} \times \cdots \times \overline{\mathcal{M}_{0,l_{k+1}+3}}$$

with $l_1 + l_2 + \cdots + l_{k+1} = n - 3 - k$ (see [BFLS99]). Then the corollary is a direct consequence of Künneth formula and Theorem 3.5. \square

3.3. The motive of $\mathcal{M}_{0,n}$. The motives of the open moduli spaces of curve $\mathcal{M}_{0,n}$ are mixed Tate and satisfy the (BS) property. This is proved in the following section.

First we recall some facts about the boundary of $\overline{\mathcal{M}_{0,n}}$ and its stratified structure. It has already been remarked that $\partial\overline{\mathcal{M}_{0,n}} = \overline{\mathcal{M}_{0,n}} \setminus \mathcal{M}_{0,n}$ is a normal crossing divisor (cf. [Knu83, Theorem. 2.7]). Let \tilde{S} be the intersection of k irreducible codimension 1 components of $\partial\overline{\mathcal{M}_{0,n}}$. Then \tilde{S} is isomorphic to the product of $k+1$ moduli spaces of curves

$$\tilde{S} \simeq \overline{\mathcal{M}_{0,n_1}} \times \cdots \times \overline{\mathcal{M}_{0,n_{k+1}}}$$

such that $\sum_{i=1}^{k+1} (n_i - 3) = n - 3 - k$.

Writing $\partial\overline{\mathcal{M}_{0,n}}$ as the union of its irreducible components

$$\partial\overline{\mathcal{M}_{0,n}} = \bigcup_{i=1}^N D_i,$$

one can assume that $\bar{S} = \cup_{i=1}^k D_i$. The open strata \mathring{S} is defined as

$$\mathring{S} = \bar{S} \setminus \left(S \cap \left(\bigcup_{i=k+1}^N D_i \right) \right)$$

and is isomorphic to

$$\bar{S} \simeq \mathcal{M}_{0,n_1} \times \cdots \times \mathcal{M}_{0,n_{k+1}}.$$

Theorem 3.8. *Let n be an integer greater or equal to 3. The motive $M_S(\mathcal{M}_{0,n})$ is in $\mathrm{DMT}_{/S,\mathbb{Z}}(S)$ the triangulated category of mixed Tate motives. Moreover the motive $M_S(\mathcal{M}_{0,n})$ satisfies (BS); that is one has*

$$\mathrm{Hom}_{\mathrm{SH}(S)}(\Sigma^\infty(\mathcal{M}_{0,n_+}), \mathbb{M}\mathbb{Z}_S(p)[k]) = \mathrm{Hom}_{\mathrm{DM}(S)}(M_S(\mathcal{M}_{0,n}), \mathbb{Z}_S(p)[k]) = 0$$

for all $p \geq 0$ and $k < 0$ and for all $p > 0$ and $k = 0$.

This statement holds in a more general situation. Let X_0 be a smooth projective scheme over S whose motive $M_S(X_0)$ is in $\mathrm{DMT}_{/S,\mathbb{Z}}(S)$ and satisfies (BS). Let $Z_0 = \cup_{i=1}^l Z_i$ a strict normal crossing divisor of X_0 . Assume that any irreducible components of any intersection of the Z_i 's has a motives in $\mathrm{DMT}_{/S,\mathbb{Z}}(S)$ and satisfies (BS). Let $U = X_0 \setminus Z_0$.

Theorem 3.9. *$M_S(U)$ is in $\mathrm{DMT}_{/S,\mathbb{Z}}(S)$ and satisfies (BS)*

Proof. The proof is a double induction on the dimension n of X_0 and l .

Let $Z' = \cup_{i=1}^{l-1} Z_i$, $X = X_0 \setminus Z'$. The intersection $Z = Z_l \cap X$ is of codimension $d = 1$. The Gysin triangle (1) insures that $M_S(U)$ sits in the distinguished triangle

$$\longrightarrow M_S(U) \longrightarrow M_S(X) \longrightarrow M_S(Z)(d)[2d] \longrightarrow M_S(U)[1] \longrightarrow \cdots$$

Applying the $\mathrm{Hom}_{\mathrm{DM}(S)}$ functor, one gets an exact sequence

$$\begin{aligned} \mathrm{H}_{\mathrm{mot}}^k(X, p) &\longrightarrow \mathrm{H}_{\mathrm{mot}}^k(U, p) = \mathrm{Hom}_{\mathrm{DM}(S)}(M(U)[1], \mathbb{Z}_S(p)[k+1]) \\ &\longrightarrow \mathrm{Hom}_{\mathrm{DM}(S)}(M_S(Z)(d)[2d], \mathbb{Z}_S(p)[k+1]) = \mathrm{H}_{\mathrm{mot}}^{k+1-2d}(Z, p-n). \end{aligned}$$

When $n = 1$ and $l = 1$ $X = X_0$ and $Z = Z_1 = Z_0$. Hence both are in $\mathrm{DMT}_{/S,\mathbb{Z}}(S)$ and satisfy (BS) which implies the theorem for U .

When $n > 1$ or $k > 1$, by induction X is in $\mathrm{DMT}_{/S,\mathbb{Z}}(S)$ and satisfies (BS). $Z = Z_l \cap X$ is equal to

$$Z = Z_l \setminus \left(\bigcup_{i=1}^{l-1} Z_i \cap Z_l \right).$$

By induction, Z is in $\mathrm{DMT}_{/S,\mathbb{Z}}(S)$ and satisfies (BS). Thus, the above exact sequence, induced by the Gysin triangle, implies the theorem for U . \square

Proof of Theorem 3.8. We apply Theorem 3.9 to the case where $X = \overline{\mathcal{M}_{0,n}}$ and $Z_0 = \partial \overline{\mathcal{M}_{0,n}}$. In this case, Theorem 3.5 and Corollary 3.7 insure that the hypothesis are satisfied. Note that in this case, Z is an open codimension 1 stratum of the compactification, hence it is isomorphic to a product of open moduli spaces of curves. One could do the induction only for the moduli spaces of curves case. \square

Remark 3.10. • Note that any strict normal crossing divisor Z_0 of X_0 induces a stratification of X_0 where the strata are given by irreducible components of the intersection of the Z_i 's. The above theorem remains valid under the same hypothesis when $U = X \setminus (\cup_{i \in I} \bar{\mathcal{S}}_i)$ is the complement of a union of closed strata defined by the divisor Z_0 ; where the strata $\bar{\mathcal{S}}_i$ have maximal dimension d_i and I is minimal a description of U . This remove some ambiguities in the choices of the strata. In this case, the proof goes by induction on the dimension of X_0 , $d = \max(d_i)$ and the number k of strata

of dimension d . As above it follows from the Gysin triangle and the long exact sequence for the H_{mot}^k . Note that closed strata of dimension 0 are disjoint and that open strata (i.e. closed strata minus closed strata of lower dimension) of dimension d are disjoint.

- Duality and Gysin morphism as explained in Section 2, and given by F. Déglise works [Dég08], gives relative motivic cohomology $H_{mot}^k(X \setminus A; B)$ where X is smooth projective, and A and B are two strict normal crossing divisors sharing no common irreducible components. This is explained by M. Levine in [Lev98].
- The theorem also shows that for any dihedral structure δ on $\{1, \dots, n\}$ the motive $M_S(\mathcal{M}_{0,n}^\delta)$ of $\mathcal{M}_{0,n}^\delta$ (which was defined by F. Brown in [Bro09, Section 2.2]) is also mixed Tate; that is a motive in $DMT_{/S, \mathbb{Z}}(S)$; and satisfies the (BS) property.

4. A MOTIVIC GROTHENDIECK-TEICHMÜLLER GROUP

This section defines a “derived” integral motivic Grothendieck-Teichmüller group over \mathbb{Z} : $GT_{\mathbb{Z}}^{mot}(\mathbb{Z})$. Here “derived” refers to the fact that the construction is based on the triangulated category $DMT_{/S, \mathbb{Z}}(S)$. M. Spitzweck’s work gives for any $n \geq 3$ an equivalence between $DMT_{/S, \mathbb{Z}}(S)$ and the perfect representations of a “derived group” $G^\bullet(\mathcal{M}_{0,n})$. These groups sit as middle terms in short exact sequences relating $G^\bullet(\mathcal{M}_{0,n})$, $G^\bullet(\mathcal{M}_{0,3}) = G^\bullet(\text{Spec}(\mathbb{Z}))$ and a “geometric part” K_n^\bullet . These exact sequences are compatible with the natural morphisms in the tower of the $\mathcal{M}_{0,n}$ (forgetting marked points and “embedding of codimension 1 component”).

$GT_{\mathbb{Z}}^{mot}(\mathbb{Z})$ is then defined as the automorphism of the tower given by the K_n^\bullet . A non derived version will be presented in the next section using rational coefficients where the t -structure is available. In this rational context and working over the spectrum of a number field, M. Levine’s work [Lev10] identifies the non derived part of K_n^\bullet with Deligne-Goncharov motivic fundamental group $\pi_1^{mot}(\mathcal{M}_{0,n})$ [DG05]. Hence the description of K_n^\bullet as a “geometric part”.

4.1. Tangential based points and normal bundle. This section describes the last requirement for developing a motivic Grothendieck-Teichmüller construction:

- A natural functor $DMT_{/S, \mathbb{Z}}(X \setminus Z) \longrightarrow DMT_{/S, \mathbb{Z}}(N_Z^0) \longrightarrow DMT_{/S, \mathbb{Z}}(Z)$ where N_Z^0 denotes the normal bundle of Z in X minus its zero section. This functor allows us to obtain a derived group morphism $G(\mathcal{M}_{0,k}) \times G(\mathcal{M}_{0,n}) \longrightarrow G(\mathcal{M}_{0,n})$ induced by the inclusion of an irreducible component $D \simeq \overline{\mathcal{M}_{0,k}} \times \overline{\mathcal{M}_{0,l}}$ of $\overline{\mathcal{M}_{0,n}}$ into $\overline{\mathcal{M}_{0,n}}$. This is a motivic version of the morphisms between fundamental groups presented in [BFLS99] in the topological context or in [Nak96] in the étale case.
- Motivic *tangential base points* or *motivic base points at infinity*. In general based points provide an augmentation to differential graded (E_∞) algebras underlying the description of mixed Tate categories as comodule categories, more specifically when one deals with the relative situation [Lev10]. They also give sections in the (derived) group setting to the morphism $p^* : DMT_{/S, \mathbb{Z}}(S) \longrightarrow DMT_{/S, \mathbb{Z}}(X)$ induced by the structural morphism $p : X \rightarrow S$. Tangential based points are used to compensate the lack of S -points (such as in the case of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ over $\text{Spec}(\mathbb{Z})$) and to preserve symmetries.

Let n be an integer greater or equal to 4. Let D be an irreducible component of $\partial\mathcal{M}_{0,n}$ and D_0 its open strata (cf. Section 3.1). The open strata is isomorphic to

$$D_0 \sim \mathcal{M}_{0,n_1} \times \mathcal{M}_{0,n_2}$$

with $n_1 + n_2 = n + 2$.

Proposition 4.1. *There is a natural functor*

$$\mathcal{L}_{D, \mathcal{M}_{0,n}}^{DM} : \mathrm{DM}_{\mathbb{Z}}(\mathcal{M}_{0,n}) \longrightarrow \mathrm{DM}_{\mathbb{Z}}(D_0)$$

sending Tate object to Tate object and hence inducing a natural functor

$$\mathcal{L}_{D, \mathcal{M}_{0,n}} : \mathrm{DMT}_{/S, \mathbb{Z}}(\mathcal{M}_{0,n}) \longrightarrow \mathrm{DMT}_{/S, \mathbb{Z}}(D_0)$$

Moreover its composition with the “structural functor” $p^ : \mathrm{DMT}_{/S, \mathbb{Z}}(S) \longrightarrow \mathrm{DMT}_{/S, \mathbb{Z}}(\mathcal{M}_{0,n})$ is isomorphic to $p_{D_0}^* : \mathrm{DMT}_{/S, \mathbb{Z}}(S) \longrightarrow \mathrm{DMT}_{/S, \mathbb{Z}}(D_0)$.*

Proof. Let X be a smooth scheme over S , $Z \xrightarrow{i} X$ a regular closed embedding such that Z is smooth over S . In our case we can also assume that Z is a divisor of X . Let X^0 be the open complement and N_Z^0 the normal bundle of Z in X with zero section removed. In [Spi01], M. Spitzweck defined, as a consequence of his Proposition 15.19, a natural functor

$$\mathcal{L}_{X, Z}^{DM} : \mathrm{DM}_{\mathbb{Z}}(X^0) \longrightarrow \mathrm{DM}_{\mathbb{Z}}(N_Z^0).$$

Then, we will apply this functor to the situation $X = \mathcal{M}_{0,n} \cup D_0$ and compose it with the pull-back functor induced by an everywhere non-zero S -section $\sigma : D_0 \longrightarrow N_{D_0}^0$ given by Proposition 3.2. In order to show that it sends Tate objects to Tate objects, we need to enter a little in the construction of the functor $\mathcal{L}_{X, Z}$ which relies for its geometric part on the (affine) deformation to the normal cone.

The deformation to the normal cone is a key geometrical construction need in the studies of Gysin maps and was explicitly used and formalized by W. Fulton [Ful98]. Later on, it plays an important role in defining *specialization maps* for example in microlocal theory of sheave [KS94], and was developed and generalized in [Ros96] and [Ivo14] to higher deformations in order to study the A_{∞} structure of cycle modules.

The starting situation is the following :

$$(4) \quad \begin{array}{ccccc} Z & \xrightarrow{i_Z} & X & \xleftarrow{j_{X^0}} & X^0 = X \setminus Z \\ & \searrow p_Z & \downarrow p_X & \swarrow p_{X^0} & \\ & & S & & \end{array}$$

Then, one considers the blow-up $\mathrm{Bl}_{Z \times \{0\}} X \times \mathbb{A}_S^1$ of $X \times \mathbb{A}_S^1$ along $Z \times \{0\}$:

$$\pi : \mathrm{Bl}_{Z \times \{0\}} X \times \mathbb{A}_S^1 \longrightarrow X \times \mathbb{A}_S^1.$$

Its restriction to $X \times \mathbb{G}_{m, S}$ is by definition isomorphic to $X \times \mathbb{G}_{m, S}$ while its restriction to $X \times \{0\}$ has two components, one of them is $\mathrm{Bl}_Z X$ and the other one is $\mathbb{P}(N_Z \oplus \mathcal{O}_Z)$ where N_Z denotes the normal bundle of Z in X . They intersect each other on $\mathbb{P}(N_Z)$ which is the exceptional divisor of $\mathrm{Bl}_Z X$. The deformation of Z to the normal cone is defined as

$$D(X, Z) = (\mathrm{Bl}_{Z \times \{0\}} X \times \mathbb{A}_S^1) \setminus \mathrm{Bl}_Z X.$$

In terms of spectrum, if \mathcal{I}_Z denotes the sheaf of ideal defining Z , the deformation $D(X, Z)$ is given by $\mathrm{Spec}(\mathcal{A}_{X, Z})$ where

$$\mathcal{A}_{X, Z} = \bigoplus_{n \in \mathbb{Z}} \mathcal{I}_Z^n t^{-n} \subset \mathcal{O}_X[t, t^{-1}]$$

with the convention that $\mathcal{I}_Z^n = \mathcal{O}_X$ as soon as $n \leq 0$. Dropping the subscript S when the situation is clear enough, the geometric situation is described by the

following diagram

(5)

$$\begin{array}{ccccccc}
 & & \xrightarrow{s_0} & & & & \\
 Z & \xleftarrow{p_{N_Z}} & N_Z & \xrightarrow{i_{N_Z}} & D(X, Z) & \xleftarrow{j_{X \times \mathbb{G}_m}} & X \times \mathbb{G}_m \xrightarrow{p \rightarrow X} X \\
 \downarrow \sim & \searrow p_{N_Z} & \downarrow f|_{\{0\}} & & \downarrow f & \searrow \pi|_{D(X, Z)} & \downarrow = \\
 Z \times \{0\} & & & & & & X \times \mathbb{G}_m \\
 & \searrow & & & \swarrow & & \downarrow f|_{\mathbb{G}_m} \\
 & & \{0\} & \xrightarrow{i_0} & \mathbb{A}^1 & \xleftarrow{j_{\mathbb{G}_m}} & \mathbb{G}_m
 \end{array}$$

In the above diagram the two “big rectangles” are Cartesian. Note that the map f and hence maps $f|_{\{0\}}$ and $f|_{\mathbb{G}_m}$ are smooth because Z itself is smooth over S . This was remarked by J. Ayoub in [Ayo07a, Beginning of Section 1.6.1 after diagram (1.37)].

The open deformation $D^0(X, Z)$ is obtain by removing in $D(X, Z)$ the strict transform of $Z \times \mathbb{A}^1$; that is the closure in $D(X, Z)$ of $Z \times \mathbb{G}_m$. The properties of $D^0(X, Z)$ are resumed in the following Cartesian diagram

$$\begin{array}{ccccccc}
 Z & \xleftarrow{p_{N_Z}^0} & N_Z^0 & \xrightarrow{i_{N_Z^0}} & D^0(X, Z) & \xleftarrow{j_{X^0 \times \mathbb{G}_m}} & X^0 \times \mathbb{G}_m \xrightarrow{p \rightarrow X^0} X^0 \\
 & & \downarrow f|_{\{0\}} & & \downarrow f & & \downarrow p \rightarrow \mathbb{G}_m \\
 & & \{0\} & \xrightarrow{i_{\{0\}}} & \mathbb{A}^1 & \xleftarrow{j_{\mathbb{G}_m}} & \mathbb{G}_m
 \end{array}$$

(6)

which is “an open immersion” of the previous one with closed complement given over $\{0\}$ and \mathbb{G}_m by $s_0(Z)$ and Z respectively.

From this geometric situation M. Spitzweck obtains [Spi01, Proposition 15.19] an isomorphism

$$(7) \quad i_Z^* j_{X^0}^* \mathcal{M}\mathbb{Z}_{X^0} \simeq p_{N_Z}^0 \mathcal{M}\mathbb{Z}_{N_Z^0}$$

by comparing the inclusions of $X \xrightarrow{\sim} X \times \{0\}$ and $X \xrightarrow{\sim} X \times \{1\}$ in the strict transform of $X \times \mathbb{A}^1$ in $D(X, Y)$ and similarly for Z .

Then M. Spitzweck uses one of his main results [Spi01, Corollary 15.14] to identify the homotopy category of modules over $p_{N_Z}^0 \mathcal{M}\mathbb{Z}_{N_Z^0}$ with the full triangulated subcategory of $\mathrm{DM}_{\mathbb{Z}}(S)$ generated by homotopy colimits of pull-back by $p_{N_Z}^0$ of objects from $\mathrm{DM}_{\mathbb{Z}}(Z)$. The composition of $\mathrm{DM}_{\mathbb{Z}}(X^0) \rightarrow \mathcal{H}(i_Z^* j_{X^0}^* - \mathrm{Mod})$ with the two identifications gives a functor

$$\mathcal{L}_{X, Z}^{DM} : \mathrm{DM}_{\mathbb{Z}}(X^0) \longrightarrow \mathrm{DM}_{\mathbb{Z}}(N_Z^0).$$

Let $p_{X^0} : X^0 \rightarrow S$ be the structural morphism. The composition of $p_{X^0}^* : \mathrm{DM}_{\mathbb{Z}}(S) \rightarrow \mathrm{DM}_{\mathbb{Z}}(X^0)$ with $\mathcal{L}_{X, Z}$ is isomorphic to $\mathrm{DM}_{\mathbb{Z}}(S) \rightarrow \mathrm{DM}_{\mathbb{Z}}(N_Z^0)$ induced by the structural morphism of N_Z^0 because

- the compatibility with objects lifted from the base is given at the end of Corollary 15.14 in [Spi01];
- the condition of corollary 15.14, asking that $M \otimes p_{N_Z}^0 \mathcal{M}\mathbb{Z}_{N_Z^0}$ is isomorphic to $p_{N_Z}^0 p_{N_Z}^{0*}(M)$ for any M in $\mathrm{DM}_{\mathbb{Z}}(Z)$, is satisfied.

Note that these two same properties insure that $\mathcal{L}_{X,Z}$ sends Tate object to Tate object.

Note that this material has also been developed in [Spi05] with some more details. \square

Remark 4.2. We develop in this remark another approach to limit motives : the nearby cycle functor [Ayo07b]. This method has been used by J. Ayoub in [Ayoa, Ayob] in the case of curve over field. The following construction agrees with M. Spitzweck's one; see [Spi05, Spi01]. We explain below how deformation to the normal cone allows us to obtain a limit motive functor from Ayoub's nearby cycle functor. For the rest of this remark, we assume that Ayoub's formalism is available ; that is we assume that the functor $DM_{/S,R}()$ associating to $X \in \text{Sm}_S$ the triangulated category $DM_{/S,R}(X)$ (using R coefficients) is coming from a monoidal stable homotopic algebraic derivator on diagrams of quasi-projective scheme over S . This assumption applies directly to our situation when working with rational coefficients ($R = \mathbb{Q}$) and the Beilinson's E_∞ -ring spectrum $M\mathbb{Q}_X$ as in section 5.1 below ($S = \text{Spec}(\mathbb{Z})$ is omitted from the notation).

We give again diagram (6) :

$$\begin{array}{ccccc}
 N_Z^0 & \xrightarrow{i_{N_Z^0}} & D^0(X, Z) & \xleftarrow{j_{X^0 \times \mathbb{G}_m}} & X^0 \times \mathbb{G}_m & \xrightarrow{p \rightarrow X^0} & X^0 \\
 \downarrow f|_{\{0\}} & & \downarrow f & & \downarrow p \rightarrow \mathbb{G}_m & & \\
 \{0\} & \xrightarrow{i_{\{0\}}} & \mathbb{A}_S^1 & \xleftarrow{j_{\mathbb{G}_m}} & \mathbb{G}_{mS} & & \\
 & \searrow = & \downarrow & & \swarrow q_{\mathbb{G}_m} & & \\
 & & S & & & &
 \end{array}$$

which corresponds to the situation of a “specialization functor” over the base \mathbb{A}^1 as described by Ayoub [Ayo07b, Section 3.1 ; 3.4 and 3.5]. The nearby cycles functor from Ayoub gives us

$$\Psi_f : DM_R(X \times \mathbb{G}_m) \longrightarrow DM_R(N_Z^0).$$

The limit motives functor is then obtained by composing with $p^*_{\rightarrow X^0}$:

$$\mathcal{L}_{X,Z}^{DM\Psi} : DM_R(X^0) \xrightarrow{p^*_{\rightarrow X^0}} DM_R(X \times \mathbb{G}_m) \xrightarrow{\Psi_f} DM_R(N_Z^0).$$

In the case $X^0 = \mathcal{M}_{0,n}$ and $Z = D_0$, we compose as previously this functor by the non zero section of $p_{N_Z^0}^0 : N_Z^0 \rightarrow Z$ given by Proposition 3.2 and obtain

$$\mathcal{L}_{D,\mathcal{M}_{0,n}}^{DM\Psi} : DM_R(\mathcal{M}_{0,n}) \longrightarrow DM_R(D_0).$$

The compatibility with mixed Tate categories follows from the following facts:

- (1) Tate objects in $DM_R(X^0)$; denoted by $R_{X^0}(i)$ in this remark; are lifted from the ones in $DM_R(S)$. Hence their pull-back in $DM_R(X^0 \times \mathbb{G}_m)$ can be seen either as lifted from \mathbb{G}_{mS} or as lifted from S .
- (2) In the first case, by the compatibility of specialization functor with smooth morphisms [Ayo07b, Definition 3.1.1] insures that

$$\Psi_f(p^*_{\rightarrow \mathbb{G}_m} R_{\mathbb{G}_m}(i)) \simeq f|_{\{0\}}^* \circ \Psi_{\text{id}_{\mathbb{A}^1}}(R_{\mathbb{G}_m}(i)).$$

Now, seeing the Tate objects as lifted from S by $q_{\mathbb{G}_m}^*$, we can apply Proposition 3.5.10 in [Ayo07b] which insures that

$$\Psi_{\text{id}_{\mathbb{A}^1}} \circ q_{\mathbb{G}_m}^* \sim \text{id}.$$

Hence the functor

$$\mathcal{L}_{X,Z}^{DM\Psi} : \text{DM}_R(X^0) \longrightarrow \text{DM}_R(N_Z^0)$$

sends Tate object to Tate objects (eventually composing with the natural isomorphic transformation). It also insures that its composition with

$$p_{X^0} : \text{DM}_R(S) \longrightarrow \text{DM}_R(X^0)$$

equals $\text{DM}_R(S) \longrightarrow \text{DM}_R(N_Z^0)$ induced by the structural morphism of N_Z^0 .

This gives us the desired functor on mixed Tate categories

$$\mathcal{L}_{D,\mathcal{M}_{0,n}}^{DM\Psi} : \text{DM}_R(\mathcal{M}_{0,n}) \longrightarrow \text{DM}_R(D_0).$$

We come now to the more delicate aspect of tangential based point in general situation. The general situation is the following : $X \xrightarrow{p_X} S$ a smooth scheme with a strict normal crossing divisor $Z = \cup_{i \in I} Z_i$. We will denote by Z_J the intersection $\cap_{i \in J} Z_i$. Note that the Z_J are also smooth over S . Note that in our applications; that is $X = \overline{\mathcal{M}_{0,n}}$ and $Z = \partial \overline{\mathcal{M}_{0,n}}$; the Z_J 's are irreducible and we will assume it is the case for the following description. If it is not the case, the description below works as well with an extra care given to the various irreducible components of the Z_J .

As previously let X^0 denote $X \setminus Z$. Let J be a subset of I and let Z_J^0 be the “open stratum”

$$Z_J \setminus (\cup_{i \in I \setminus J} Z_i \cap Z_J).$$

Let N_i (resp. N_i^0) denote the normal bundle of Z_i in X (resp. with zero section removed), N_J (resp. N_J^0) is defined as the fiber product of the $N_j|_{Z_J}$ (resp. $N_j^0|_{Z_J}$) over Z_J and N_{J^0} (resp. $N_{J^0}^0$) its restriction to Z_J^0 .

We are interested in generalizing the previous situation and having a functor

$$\mathcal{L}_{X,J} : \text{DM}(X^0) \longrightarrow \text{DM}(N_{J^0}^0)$$

compatible with mixed Tate categories and structural pull-back functors. We want then to apply this functor in the case where Z_J is an S -point, that is of maximal codimension.

First of all remark that by setting

$$X' = X \setminus (\cup_{i \in I \setminus J} Z_i)$$

we can assume that $I = J$. In this case Z_J^0 (resp. N_{J^0} , $N_{J^0}^0$) is simply Z_J (resp. N_J , N_J^0). We treat below only this situation.

Note also that in our strict normal crossing divisor situation N_J equals N_{Z_J} the normal bundle of Z_J in X . Moreover locally with affine coordinates, or when N_{Z_J} is trivial, one has a isomorphism between N_J^0 and $(\mathbb{G}_{m,Z_J})^{|J|}$.

Lemma 4.3 (Consequence of [Spi01, Proposition 15.22]). *There is a natural functor*

$$\mathcal{L}_{X,J}^{DM} : \text{DM}_{\mathbb{Z}}(X^0) \longrightarrow \text{DM}_{\mathbb{Z}}(N_J^0)$$

preserving Tate objects and compatible with structural pull-back morphisms. Hence we obtain a functor between mixed Tate categories

$$\mathcal{L}_{X,J}^{DM} : \text{DMT}_{/S,\mathbb{Z}}(X^0) \longrightarrow \text{DMT}_{/S,\mathbb{Z}}(N_J^0)$$

such that its composition with $\text{DMT}_{/S,\mathbb{Z}}(S) \longrightarrow \text{DMT}_{/S,\mathbb{Z}}(X^0)$ equals the functor

$$\text{DMT}_{/S,\mathbb{Z}}(S) \longrightarrow \text{DMT}_{/S,\mathbb{Z}}(N_J^0)$$

induced by the structural morphism of N_J^0 .

Comments on the construction. The proof consists in obtaining the generalization of the isomorphism 7

$$i_J^* j_{X^0}^* \mathbb{M}\mathbb{Z}_{X_0} \simeq p_{N_J}^0 \mathbb{M}\mathbb{Z}_{N_J^0}$$

where i_J denotes the regular embedding $Z_J \rightarrow X$.

This isomorphism is obtained by taking the fiber product over $X \times \mathbb{A}^1$ of the deformation $D(X, Z_j)$ (resp. $D^0(X, Z_j)$) for all Z_j . Then his proof goes mostly as in the case where there is only one Z_j by remarking that $\mathbb{M}\mathbb{Z}_{X_0}$ is the pull-back from $\mathbb{M}\mathbb{Z}_S$ by $(p_X \circ j_{X^0})^*$.

As previously, the compatibility with mixed Tate object and pull-back by structural morphism rely partly on [Spi01, Corollary 15.14]. \square

Remark 4.4 (on higher deformations to the normal cone and nearby cycle functor). As previously; especially when working with the Beilinson spectrum, that is working with rational coefficient; one might prefer using Ayoub's nearby cycles functor. We introduce now a higher deformation to the normal cone as presented in [Ivo14, Section 3.1.3] following M. Rost in [Ros96, §(10.6)]. Let \mathcal{J}_j denote the sheaf of ideal defining the Z_j 's and let k denote the cardinal of J (we treat only the case $J = I$). We assume that that $J = \{1, \dots, k\}$ as it induces an easier notation. Then the subalgebra of $\mathcal{O}_X[t_1, t_1^{-1}, \dots, t_k, t_k^{-1}]$

$$\mathcal{A}_{X,J} = \bigoplus_{(a_1, \dots, a_k) \in \mathbb{Z}^k} \mathcal{J}_1^{a_1} \cdots \mathcal{J}_k^{a_k} t_1^{a_1} \cdots t_k^{a_k}$$

is quasi-coherent over $\mathcal{O}_X[t_1, \dots, t_n]$; in the above definition, as previously, $\mathcal{J}_j^{a_j} = \mathcal{O}_X$ as soon as $a_j \leq 0$. The simultaneous deformation of the Y_j 's is defined as :

$$(8) \quad D(X; Y_1, \dots, Y_k) = D(X; J) = \text{Spec}(\mathcal{A}_{X,J}).$$

Inverting the t_j , one obtains

$$\mathcal{A}_{X,J}[t_1^{-1}, \dots, t_k^{-1}] = \mathcal{O}_X[t_1, t_1^{-1}, \dots, t_k, t_k^{-1}]$$

and hence a canonical isomorphism between $X \times \mathbb{G}_m^k$ and the restriction of $D(X, J)$ over \mathbb{G}_m^k and the following diagram

$$(9) \quad \begin{array}{ccccc} D(X, Z) & \xleftarrow{j_D|_{\mathbb{G}_m^k}} & D(X, J)|_{\mathbb{G}_m^k} & & X \\ & \searrow & \downarrow f_J|_{\mathbb{G}_m^k} & \searrow = & \uparrow p_{\rightarrow X} \\ & & X \times \mathbb{A}^k & \xleftarrow{\quad} & X \times \mathbb{G}_m^k \\ & \swarrow f_J & & \swarrow p_{\rightarrow \mathbb{G}_m^k} & \\ \mathbb{A}^k & \xleftarrow{j_{\mathbb{G}_m^k}} & \mathbb{G}_m^k & & \end{array}$$

where the square and the parallelograms are Cartesian.

Note that the construction is compatible with permutation of the coordinates on \mathbb{A}^k and permutation of the Y_j 's. Inverting only t_1, \dots, t_l ($l < k$), one obtains

$$D(X, J)|_{\mathbb{G}_m^l \times \mathbb{A}^{k-l}} = \mathbb{G}_m^l \times D(X; Y_{l+1}, \dots, Y_k).$$

F. Ivorra has described in [Ivo14] the fiber over $t_1 = \dots = t_l = 0$. The description goes by induction on l . For $l = 1$, $Y_{[l]}$ is simply Y_1 , $N_{[l]}$ is N_1 the normal bundle of Y_1 in X . For $l \geq 2$, let $Y_{[l]}$ be the intersection

$$Y_1 \cap \dots \cap Y_l$$

and $N_{[l]}$ is the normal bundle of $N_{[l-1]}|_{Y_{[l]}}$ in $N_{[l-1]}$ which can be written as

$$N_{[l]} = N(N_{[l-1]}, N_{[l-1]}|_{Y_{[l]}}).$$

Now let $D^{[l]}$ be the deformation

$$D(N_{[l]}; N_{[l]}|_{Y_{l+1} \cap Y_{[l]}}, \dots, N_{[l]}|_{Y_k \cap Y_{[l]}}).$$

Then one has

$$D(X, J)|_{t_1=\dots=t_l=0} = D^{[l]}.$$

The fiber over $(0, \dots, 0)$; that is when all the t_j 's are zero; is isomorphic to N_J the normal bundle of Y_J in X . As a last remark, the description of $\mathcal{A}_{X,J}$ shows that the restriction of $D(X; J)$ to the diagonal (or any line going through the origin) is isomorphic to $D(X; Y_J)$.

Now, as in the case of a simple deformation, one can remove the strict transform of $Z \times \mathbb{A}^k$ in $D(X, J)$ and obtain an "open deformation" $D^0(X, J)$ whose restriction to \mathbb{G}_m^k is simply $X^0 \times \mathbb{G}_m^k$. Its fiber over $(0, \dots, 0)$ is isomorphic to N_J^0 and its restriction to the diagonal Δ_k gives us

$$\begin{array}{ccccc} N_J^0 & \xrightarrow{i_{N_Z^0}} & D^0(X, J)|_{\Delta_k} & \xleftarrow{j_{X^0 \times \mathbb{G}_m}} & X^0 \times \mathbb{G}_m & \xrightarrow{p \rightarrow X^0} & X^0 \\ \downarrow f_J|_{\{0\}} & & \downarrow f_J & & \downarrow p \rightarrow \mathbb{G}_m & & \\ \{0\} & \xrightarrow{i_{\{0\}}} & \mathbb{A}_S^1 & \xleftarrow{j_{\mathbb{G}_m}} & \mathbb{G}_m & & \\ & \searrow = & \downarrow & \swarrow q_{\mathbb{G}_m} & & & \\ & & S & & & & \end{array}$$

And we proceed as in Remark 4.2 in order to obtain a functor

$$\mathcal{L}_{X,J}^{DM,\Psi} : \mathrm{DM}(X^0) \xrightarrow{p^* \rightarrow X^0} \mathrm{DM}(X^0 \times \mathbb{G}_m) \xrightarrow{\Psi_{f_J}} \mathrm{DM}(N_J^0).$$

Compatibilities with mixed Tate categories and pull-back by structural morphisms are as in Remark 4.2.

Conjecture 1. The geometric situation given by the higher deformation $D(X, J)$ makes it possible to use a succession of specialization functor each corresponding to an \mathbb{A}^1 factor. This procedure does not depend on choices when considering only the mixed Tate motives categories. It agrees with our construction using the diagonal.

Now, we can apply the above functor to our particular situation in order to obtain *base point at infinity* or *tangential base point*. Let $n \geq 4$. Let v be a point in $\overline{\mathcal{M}_{0,n}}$ given by a closed stratum of $\partial \overline{\mathcal{M}_{0,n}}$ of maximal codimension. The stratum v is the non-empty intersection of exactly $n-3 = \dim_S(\overline{\mathcal{M}_{0,n}})$ irreducible component of $\partial \overline{\mathcal{M}_{0,n}}$ of codimension 1

$$v = \bigcap_{\substack{D \text{ cl. str. of } \partial \overline{\mathcal{M}_{0,n}} \\ \mathrm{codim}(D)=1 \\ v \in D}} D = \bigcap_{j \in J} D_j$$

where $J = \{1, \dots, n-3\}$ corresponds to a numbering of the closed codimension 1 strata D with $v \in D$. The normal bundle N_v of v in $\overline{\mathcal{M}_{0,n}}$ is trivial.

Definition 4.5. A *tangential base point* x_v of $\mathcal{M}_{0,n}$ is the choice of a closed strata v of maximal codimension in $\overline{\partial\mathcal{M}_{0,n}}$ and of a non zero S -point in $N_v^0 = N_J^0$ with the notation of Lemma 4.3 (note that in this case $N_{J_0}^0 = N_J^0$ as $v = Z_J$ can not have a non empty intersection with any other component of $\overline{\partial\mathcal{M}_{0,n}}$).

Proposition 4.6. *For any tangential base point x_v of $\mathcal{M}_{0,n}$, there is a natural functor*

$$\tilde{x}_v^{DM,*} : \mathrm{DM}_{\mathbb{Z}}(\mathcal{M}_{0,n}) \longrightarrow \mathrm{DM}_{\mathbb{Z}}(S)$$

sending the Tate object to the Tate object and hence inducing a natural functor

$$\tilde{x}_v^* : \mathrm{DMT}_{/S,\mathbb{Z}}(\mathcal{M}_{0,n}) \longrightarrow \mathrm{DMT}_{/S,\mathbb{Z}}(S).$$

Both functors are compatible, in the sens of Lemma 4.3, with the pull-back by the structural functor from $\mathrm{DM}_{\mathbb{Z}}(S)$.

Proof. The boundary $\overline{\partial\mathcal{M}_{0,n}}$ can be written as the union of its codimension 1 irreducible component

$$\overline{\partial\mathcal{M}_{0,n}} = \bigcup_{i \in I} D_i.$$

The closed strata v defines a subset J of I by

$$v = \bigcap_{j \in J} D_j.$$

With

$$X = \overline{\mathcal{M}_{0,n}} \setminus (\cup_{i \in I \setminus J} D_i) \quad \text{and} \quad Z_j = D_j \setminus (\cup_{i \in I \setminus J} D_i) \cap D_j$$

Lemma 4.3 gives us a functor

$$\mathcal{L}_{X,J}^{DM} : \mathrm{DM}_{\mathbb{Z}}(X^0) = \mathrm{DM}_{\mathbb{Z}}(\mathcal{M}_{0,n}) \longrightarrow \mathrm{DM}_{\mathbb{Z}}(N_J^0).$$

The functor $\tilde{x}_v^{DM,*}$ is obtained by composing $\mathcal{L}_{X,J}^{DM}$ with the pull-back of the S -point in N_J^0 . In this application, we have simply $Z_J = v$ and that $Z_J^0 = Z_J$. \square

There is a canonical system of tangential based point over $\mathrm{Spec}(\mathbb{Z})$ on $\mathcal{M}_{0,n}$. A point $v = \cap_{j \in J} D_j$ of this system is given by the closed strata of maximal codimension of $\overline{\partial\mathcal{M}_{0,n}}$. In order to choose an S -point in its normal bundle, we chose a dihedral structure δ on the marked points and vertex coordinates x_j corresponding to the point v (cf. [Bro09, Definition 2.18]). Note that the chosen vertex coordinates might differ only by the choice of the numbering. These vertex coordinates induce a basis on N_J . The sum of the vector of this basis depends only on the dihedral structure δ and is the S -point in N_J^0 attached to v and δ . Changing the delta amount in introducing signs instead of taking the sums of the basis's vector (see. [Bro09, §2.7]).

Definition 4.7. Let $P_{n,\infty}$ denote the set of these tangential based points.

Remark that F. Brown developed his notion of based point at infinity for the $\mathcal{M}_{0,n}$ in relation with the question of unipotent closures and periods of the moduli space of curves in genus 0: see [Bro09] Definition 3.16 and Example 3.17 and before Section 6.3.

Remark 4.8. The results of the above subsection and of section 3 hold by the same arguments in a “more classical” motivic category as the one developed by Cisinski and Déglise [CD09] : rational coefficients over a general base and Beilinson E_{∞} -ring spectrum. Later on we will be interested in the case where the base is either a number field of the ring of integer of such a field with some prime inverted (see Section 5).

4.2. The motivic short exact sequence. Derived groups scheme were in particular studied by B. Toën in [Toe03] and Spitzweck in [Spi10]. They can be considered as spectrum of E_∞ algebras (with a coproduct structures). Now, using Spitzweck [Spi10], we define a derived group scheme associated to the category $\mathrm{DM}_{\mathbb{Z}}(\mathcal{M}_{0,n})$. Recall that $S = \mathrm{Spec}(\mathbb{Z})$. Using Theorem 2.10 and Theorem 3.8, we can apply directly Theorem 8.4 in [Spi13].

Theorem 4.9. *Let $n \geq 3$ be an integer. There is an affine derived group scheme $G_{\mathbb{Z},n}^\bullet$ over \mathbb{Z} such that*

$$\mathrm{Perf}(G_{\mathbb{Z},n}^\bullet) \simeq \mathrm{DMT}_{/S,\mathbb{Z}}^{gm}(\mathcal{M}_{0,n})$$

where Perf denotes the category of perfect representation and $\mathrm{DM}_{/S,\mathbb{Z}}^{gm}$ the full subcategory of $\mathrm{DMT}_{/S,\mathbb{Z}}(\mathcal{M}_{0,n})$ of compact objects. We shall write simply $G^\bullet(\mathbb{Z})$ for $G_{\mathbb{Z},3}^\bullet$.

Moreover the structural morphism $p_n : \mathcal{M}_{0,n} \rightarrow S = \mathrm{Spec}(\mathbb{Z})$ induces a surjective morphism

$$G_{\mathbb{Z},n}^\bullet \xrightarrow{\phi_n} G_{\mathbb{Z},3}^\bullet \longrightarrow 0.$$

induced by the natural pull-back p_n^* at the category level. Permutation of the marked points on $\mathcal{M}_{0,n}$ induces an action of the symmetric group on $G_{\mathbb{Z},n}^\bullet$.

Note that similarly, we define $G_{\mathbb{Z},k_1,k_2}^\bullet$ associated to $\mathrm{DMT}_{/S,\mathbb{Z}}^{gm}(\mathcal{M}_{0,k_1} \times \mathcal{M}_{0,k_2})$.

Defining $K_{\mathbb{Z},n}^\bullet$ as the kernel of ϕ_n , we obtain for any $n \geq 4$ a short exact sequence (SES _{n})

$$0 \longrightarrow K_{\mathbb{Z},n}^\bullet \longrightarrow G_{\mathbb{Z},n}^\bullet \longrightarrow G^\bullet(\mathbb{Z}).$$

The compatibility property with structural morphisms in Proposition 4.6 shows that this short exact sequence is split by any choice of a (tangential) S -point of $\mathcal{M}_{0,n}$. We restrict ourselves to the family of tangential based points in P_∞ .

Proposition 4.10. *Let x_v be a tangential based point $\mathcal{M}_{0,n}$ ($n \geq 4$) in $P_{n,\infty}$. Then x_v induces a splitting*

$$0 \longrightarrow K_{\mathbb{Z},n}^\bullet \longrightarrow G_{\mathbb{Z},n}^\bullet \xrightarrow[\phi_n]{\tilde{x}_v} G^\bullet(\mathbb{Z}) \longrightarrow 0$$

Proposition 4.11. *The morphisms $\psi_{n,i} : \mathcal{M}_{0,n} \rightarrow \mathcal{M}_{0,n-1}$ forgetting the i -th marked points induces a commutative diagram :*

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_{\mathbb{Z},n}^\bullet & \longrightarrow & G_{\mathbb{Z},n}^\bullet & \xrightarrow[\phi_n]{} & G^\bullet(\mathbb{Z}) \longrightarrow 0 \\ & & \downarrow \tilde{\psi}_{n,i} & & \downarrow \psi_{n,i} & & \parallel \\ 0 & \longrightarrow & K_{\mathbb{Z},n-1}^\bullet & \longrightarrow & G_{\mathbb{Z},n-1}^\bullet & \xrightarrow[\phi_{n-1}]{} & G^\bullet(\mathbb{Z}) \longrightarrow 0 \end{array}$$

where by an abuse of notation the morphisms between derived groups are denoted as the morphisms between schemes.

Proof. The functoriality of the pull-back functors makes the equivalent of the right square commutes between the categories $\mathrm{DMT}_{/S,\mathbb{Z}}^{gm}$. Considering these categories as categories of perfect representation, that is as categories of (co)modules over an E_∞ algebra, this functors induce morphisms between the affine derived group scheme

$$\psi_{n,i} : G_{\mathbb{Z},n}^\bullet \longrightarrow G_{\mathbb{Z},n-1}^\bullet.$$

compatible with ϕ_n and ϕ_{n-1} together with an induced morphism on the kernels. \square

Proposition 4.12. *Let D be an irreducible component of $\partial\overline{\mathcal{M}}_{0,n}$ and D_0 its open strata (cf. Section 3.1). D_0 is isomorphic to*

$$D_0 \sim \mathcal{M}_{0,n_1} \times \mathcal{M}_{0,n_2}$$

with $n_1 + n_2 = n + 2$.

The inclusion $i_D : \overline{\mathcal{M}}_{0,n_1} \times \overline{\mathcal{M}}_{0,n_2} \rightarrow \overline{\mathcal{M}}_{0,n}$ induces a gluing morphism :

$$i_{n_1,n_2,D} : G_{\mathbb{Z},n_1,n_2}^\bullet \rightarrow G_{\mathbb{Z},n}^\bullet$$

and

$$\tilde{i}_{n_1,n_2,D} : K_{\mathbb{Z},n_1,n_2}^\bullet \rightarrow K_{\mathbb{Z},n}^\bullet.$$

Moreover the projections $\mathcal{M}_{0,n_1} \times \mathcal{M}_{0,n_2} \rightarrow \mathcal{M}_{0,n_i}$ induce morphisms

$$p_{n_1,n_2} : G_{\mathbb{Z},n_1,n_2}^\bullet \rightarrow G_{\mathbb{Z},n_1}^\bullet \times G_{\mathbb{Z},n_2}^\bullet$$

and

$$\tilde{p}_{n_1,n_2} : K_{\mathbb{Z},n_1,n_2}^\bullet \rightarrow K_{\mathbb{Z},n_1}^\bullet \times K_{\mathbb{Z},n_2}^\bullet$$

The above morphisms makes the diagrams below commute but the bottom line is not necessarily exact

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_{\mathbb{Z},n}^\bullet & \longrightarrow & G_{\mathbb{Z},n}^\bullet & \xrightarrow{\phi_n} & G^\bullet(\mathbb{Z}) \longrightarrow 0 \\ & & \uparrow \tilde{i}_{n_1,n_2,D} & & \uparrow i_{n_1,n_2,D} & & \uparrow = \\ 0 & \longrightarrow & K_{\mathbb{Z},n_1,n_2}^\bullet & \longrightarrow & G_{\mathbb{Z},n_1,n_2}^\bullet & \xrightarrow{\phi_{n-1}} & G^\bullet(\mathbb{Z}) \longrightarrow 0 \\ & & \downarrow p_{n_1,n_2} & & \downarrow p_{n_1,n_2} & & \downarrow = \\ & & K_{\mathbb{Z},n_1}^\bullet \times K_{\mathbb{Z},n_2}^\bullet & \longrightarrow & G_{\mathbb{Z},n_1}^\bullet \times G_{\mathbb{Z},n_2}^\bullet & \longrightarrow & G^\bullet(\mathbb{Z}) \end{array}$$

As previously, by an abuse of notation, the morphisms between derived groups are denoted as the morphisms between the scheme.

Proof. First of all, Proposition 4.1 gives us a functor

$$DM(\mathcal{M}_{0,n}) \rightarrow DM(D_0) = DM(\mathcal{M}_{0,n_1} \times \mathcal{M}_{0,n_2})$$

inducing a morphism on the groups

$$G_{\mathbb{Z},n_1,n_2}^\bullet \rightarrow G_{\mathbb{Z},n}^\bullet.$$

Then, the projections $\mathcal{M}_{0,n_1} \times \mathcal{M}_{0,n_2} \rightarrow \mathcal{M}_{0,n_i}$ induce morphisms $G_{\mathbb{Z},n_1,n_2}^\bullet \rightarrow G_{\mathbb{Z},n_i}^\bullet$. \square

4.3. A motivic Grothendieck-Teichmüller group. The family of the derived groups K_n^\bullet endowed with the gluing morphism $\tilde{i}_{n_1,n_2,D}$ and forgetful morphisms $\tilde{\psi}_{n,i}$ is close to an operadic structure.

We can now define the integral derived motivic Grothendieck-Teichmüller group over \mathbb{Z} as follows.

Definition 4.13 (motivic Grothendieck-Teichmüller ($S = \text{Spec}(\mathbb{Z})$, \mathbb{Z} coefficient)). Let $GT_{\mathbb{Z}}^\bullet(S)$ be the groups of automorphisms g ; that is of auto equivalence ; of the tower of the derived groups $(K_n^\bullet)_{n \geq 4} \cup ((K_{n_1,n_2}^\bullet)_{n_1,n_2 \geq 4})$; such that g is given by two collections of morphisms $(g_n)_{n \geq 4}$ and $(g_{n_1,n_2})_{n_1,n_2 \geq 4}$ such that each g_n (resp. g_{n_1,n_2}) is an automorphism of K_n^\bullet (resp. K_{n_1,n_2}^\bullet) and the g_n 's and the g_{n_1,n_2} 's commute with the action of the symmetric group on K_n^\bullet and with morphisms $\tilde{i}_{n_1,n_2,D}$, $\tilde{p}_{n_1,n_2,D}$ and $\tilde{\psi}_{n,i}$.

Remark 4.14. It seems that in [HS00] it is only required that the morphism g_n preserves the image of morphisms $\tilde{i}_{n_1,n_2,D}$. If the approach in [HS00] is more workable, it is not as precise in its geometric implication as the one described here.

5. COMPARISON WITH CLASSICAL MOTIVIC CONSTRUCTIONS

In this section we develop how the above situation evolves in the more classical setting of rational coefficients and when working over a number field.

5.1. Rational coefficients. In this subsection we describe the situation with rational coefficients. The main advantage is that the weight t -structure is then available and allow us to use the tannakian formalism on the *non derived* category of mixed Tate motive.

In order to work with rational coefficients one can consider the Beilinson spectrum $H_{B,S}$ (see [CD09, Definition 13.1.2]) and work with the homotopy category of module over it as our derived categories of motive with \mathbb{Q} coefficients $\mathrm{DM}_{\mathbb{Q}}(S)$ (resp. $\mathrm{DM}_{\mathbb{Q}}(\mathcal{M}_{0,n})$ for $n \geq 4$). All the proof of the needed statement in the previous sections work with the same argument. Another way to work with rational coefficients is to work with the rationalization MQ_S of M. Spitzweck's spectrum MZ_S . Both approaches are equivalent thanks to Theorem 7.14 and Theorem 7.18 in [Spi13] which gives an isomorphism

$$\mathrm{MQ}_S \simeq H_{B,S};$$

and more generally, using the pull-back by the structural morphisms $\mathrm{MQ}_X \simeq H_{B,X}$.

Tate objects $\mathrm{MQ}_X(n)$ will simply be denoted by $\mathbb{Q}_X(n)$ when there is no need to insist on the spectrum they are coming from and simply by $\mathbb{Q}(n)$ when it is clear enough what X is. As previously, the derived category of mixed Tate motives $\mathrm{DMT}_{/S,\mathbb{Q}}(X)$ is defined as the full triangulated subcategory of $\mathrm{DM}_{\mathbb{Q}}(X)$ generated by Tate objects $\mathbb{Q}_X(n)$ for $n \in \mathbb{Z}$.

Hence, from the previous section, we have a family of diagrams between derived categories of mixed Tate motives with rational coefficients ($S = \mathrm{Spec}(\mathbb{Z})$)

$$(10) \quad \begin{array}{ccc} \mathrm{DMT}_{/S,\mathbb{Q}}(\mathcal{M}_{0,n-1}) & \xleftarrow{\phi_{n-1}^*} & \mathrm{DMT}_{/S,\mathbb{Q}}(S) \\ \phi_{n,i}^* \downarrow & \nearrow \tilde{x}_{v'}^* & \downarrow = \\ \mathrm{DMT}_{/S,\mathbb{Q}}(\mathcal{M}_{0,n}) & \xleftarrow{\phi_n^*} & \mathrm{DMT}_{/S,\mathbb{Q}}(S) \\ \mathcal{L}_{D,\mathcal{M}_{0,n}} \downarrow & \nearrow \tilde{x}_v^* & \downarrow = \\ \mathrm{DMT}_{/S,\mathbb{Q}}(\mathcal{M}_{0,n_1} \times \mathcal{M}_{0,n_2}) & \xleftarrow{\phi_{n_1,n_2}^*} & \mathrm{DMT}_{/S,\mathbb{Q}}(S) \end{array}$$

for any $n \geq 4$, $n_1 + n_2 = n + 2$ and D a closed codimension 1 strata of $\overline{\mathcal{M}_{0,n}}$. In the above diagram the tangential base points x_v and $x_{v'}$ are in $P_{n,\infty}$ and in $P_{n-1,\infty}$ respectively. Moreover x_v and $x_{v'}$ are compatible ; that is $\phi_{n,i}(v) = v'$ and $d\phi_{n,i}(x_v) = x_{v'}$.

In order to pass to the non derived category we need to make some comments about the t -structure defined in [Lev93] when working over a field (see also [Lev05]. As remarked in [Lev10] the arguments of [Lev93] go through provided that the Beilinson-Soulé vanishing property (BS) holds. Hence when X over S (and S) satisfies (BS) property, we obtain a tannakian category $\mathrm{MTM}_{/S}(X)$ of mixed Tate motives over X as the heart of $\mathrm{DMT}_{/S,\mathbb{Q}}$ by the t -structure (duality and tensor structure are inherited from the one in DMT). The fiber functor is induced by the weight graded piece Gr_W^* :

$$\omega : M \mapsto \bigoplus_n \mathrm{Hom}(\mathbb{Q}_X(n), \mathrm{Gr}_n^W(M)).$$

Diagram (10) is compatible with the t -structure and induces a similar diagram between tannakian category of mixed Tate motive $\text{MTM}_{/S}(-)$. The tannakian formalism allows us to identify the categories $\text{MTM}_{/S}(-)$ with the categories of graded representations of graded pro-unipotent affine algebraic group $G_{/S,n}$ and $G_{/S,n_1,n_2}$. We may drop the subscript $/S$ when the base scheme is clear enough and simply write G_n, G_{n_1,n_2} . As in the above section, $G_{/S,3}$ is denoted by $G(S)$ as it corresponds to $\text{MTM}_{/S}(S)$. Groups $G_{/S,n}$ are sometime refereed as motivic fundamental groups of $\mathcal{M}_{0,n}$. However we prefer to use the expression *tannakian groups* of $\mathcal{M}_{0,n}$ as these groups are obtained from the categories $\text{MTM}_{/S}(\mathcal{M}_{0,n})$ by the tannakian formalism. Hence diagram (10) leads to a diagram of groups

$$(11) \quad \begin{array}{ccccccc} 0 & \longrightarrow & K_{n-1} & \longrightarrow & G_{n-1} & \xrightarrow{\phi_{n-1}} & G(S) \longrightarrow 0 \\ & & \uparrow \tilde{\phi}_{n,i} & & \uparrow \phi_{n,i} & \swarrow \tilde{x}_v^* & \uparrow \tilde{x}_v^* \\ 0 & \longrightarrow & K_n & \longrightarrow & G_n & \xrightarrow{\phi_n} & G(\mathbb{Z}) \longrightarrow 0 \\ & & \uparrow \tilde{i}_{n_1,n_2,D} & & \uparrow i_{n_1,n_2,D} & \swarrow \tilde{x}_v^* & \uparrow \tilde{x}_v^* \\ 0 & \longrightarrow & K_{n_1,n_2} & \longrightarrow & G_{n_1,n_2} & \xrightarrow{\phi_{n-1}} & G(\mathbb{Z}) \longrightarrow 0 \\ & & \downarrow \tilde{p}_{n_1,n_2} & & \downarrow p_{n_1,n_2} & \swarrow \tilde{x}_v^* & \downarrow \tilde{x}_v^* \\ & & K_{n_1} \times K_{n_2} & \longrightarrow & G_{n_1} \times G_{n_2} & \longrightarrow & G(\mathbb{Z}) \end{array}$$

where the three first line are exact.

Definition 5.1 (motivic Grothendieck-Teichmüller ($S = \text{Spec}(\mathbb{Z})$, \mathbb{Q} coefficient)). Let $GT^{\text{mot}}(S)$ be the groups of automorphisms g of the tower of the groups

$$(K_n)_{n \geq 4} \cup ((K_{n_1,n_2})_{n_1,n_2 \geq 4});$$

such that g is given by two collections of morphisms $(g_n)_{n \geq 4}$ and $(g_{n_1,n_2})_{n_1,n_2 \geq 4}$ such that each g_n (resp. g_{n_1,n_2}) is an automorphism of K_n (resp. K_{n_1,n_2}) and the g_n 's and the g_{n_1,n_2} 's commute with the action of the symmetric group on K_n and with morphisms $\tilde{i}_{n_1,n_2,D}, \tilde{p}_{n_1,n_2,D}$ and $\tilde{\psi}_{n,i}$.

5.2. Working over a number field and its ring of integers. Working over the integer allows us to obtain a nice and general description. However working over a number fields would allow us to have a concrete description of the above groups or of their Hopf algebraic avatar (in the tannakian formalism) in terms of algebraic cycles as described in [Lev10]. However, before explaining this, we will take the opportunity to compare the above category of mixed Tate motives over \mathbb{Z} ($\text{MTM}_{/\text{Spec}(\mathbb{Z})}(\text{Spec}(\mathbb{Z}))$) with the one defined by Goncharov and Deligne in [DG05] which is defined as a subcategory of $\text{MTM}_{/\text{Spec}(\mathbb{Q})}(\text{Spec}(\mathbb{Q}))$.

The structural morphism $p_{\mathbb{Q}} : \text{Spec}(\mathbb{Q}) \rightarrow \text{Spec}(\mathbb{Z})$ induces a functor on the derived motivic categories

$$p_{\mathbb{Q}}^* : \text{DM}_{/\text{Spec}(\mathbb{Z}),\mathbb{Q}}(\text{Spec}(\mathbb{Z})) \rightarrow \text{DM}_{\text{Spec}(\mathbb{Z}),\mathbb{Q}}(\text{Spec}(\mathbb{Q}))$$

sending Tate object to Tate objects and compatible the t -structure. Hence it induces a functor between mixed Tate categories

$$p_{\mathbb{Q}}^* : \text{MTM}_{/\text{Spec}(\mathbb{Z})}(\text{Spec}(\mathbb{Z})) \rightarrow \text{MTM}_{/\text{Spec}(\mathbb{Z})}(\text{Spec}(\mathbb{Q})) = \text{MTM}_{/\text{Spec}(\mathbb{Q})}(\text{Spec}(\mathbb{Q})).$$

Using Remark 2.11, the same holds when \mathbb{Z} is replaced by $\mathcal{O}_{F,\mathcal{P}}$, the ring of \mathcal{P} -integers of a number field \mathbb{F} (here \mathcal{P} denotes a set of finite place of F); see [DG05,

Gon01]:

$$p_{\mathbb{F}, \mathcal{P}}^* : \text{MTM}_{/S}(S) \longrightarrow \text{MTM}_{/S}(\text{Spec}(\mathbb{F})) = \text{MTM}_{/\text{Spec}(\mathbb{F})}(\text{Spec}(\mathbb{F})).$$

where $S = \text{Spec}(\mathcal{O}_{\mathbb{F}, \mathcal{P}})$.

The functor $p_{\mathbb{F}, \mathcal{P}}$ sends the Tate object $\mathbb{Q}_S(i)$ to $\mathbb{Q}_{\mathbb{F}}(i)$ and induces on the extension groups the inclusion

$$\begin{aligned} \mathcal{O}_{\mathbb{F}, \mathcal{P}}^* \otimes \mathbb{Q} = \text{Ext}_{\text{MTM}_{/S}(S)}(\mathbb{Q}_S(0), \mathbb{Q}_S(1)) &\longrightarrow \mathbb{F}^* \otimes \mathbb{Q} = \\ &\text{Ext}_{\text{MTM}_{/\text{Spec}(\mathbb{F})}(\text{Spec}(\mathbb{F}))}(\mathbb{Q}_{\mathbb{F}}(0), \mathbb{Q}_{\mathbb{F}}(1)). \end{aligned}$$

Hence it induces an equivalence between the category $\text{MTM}_{/S}(S)$ and the category $\text{MTM}^{DG}(\mathcal{O}_{\mathbb{F}, \mathcal{P}})$ previously defined by Deligne and Goncharov (see [DG05, §1.4 and 1.7]) as the sub tannakian category of $\text{MTM}_{/\text{Spec}(\mathbb{F})}(\text{Spec}(\mathbb{F}))$ such that the coaction $\text{Ext}(\mathbb{Q}_{\mathbb{F}}(0), \mathbb{Q}_{\mathbb{F}}(1))$ on the canonical fiber functor factors through $\mathcal{O}_{\mathbb{F}, \mathcal{P}}$.

Proposition 5.2. *There is an equivalence of category*

$$\text{MTM}_{/S}(S) \simeq \text{MTM}^{DG}(\mathcal{O}_{\mathbb{F}, \mathcal{P}}).$$

Working over a number field, that is over $S = \text{Spec}(\mathbb{F})$, Theorem 3.8 also holds and the moduli spaces over curves $\mathcal{M}_{0,n}$ ($n \geq 3$) have a motive in $\text{DM}_{/S, \mathbb{Q}}(S)$ and satisfy the (BS) property. Hence Levine's works shows that the tannakian group $G_{n, \mathbb{F}}$ associated to $\text{MTM}_{/S}(\mathcal{M}_{0,n})$ is the spectrum of a Hopf algebra H_n built from algebraic cycles (see [Lev10]). More precisely, let $V_n^k(p)$ be the \mathbb{Q} vector space freely generated by closed irreducible subvarieties

$$Z \subset \mathcal{M}_{0,n} \times (\mathbb{P}^1 \setminus \{1\})^{2p-k} \times \mathbb{A}^p$$

such that the projection

$$\mathcal{M}_{0,n} \times (\mathbb{P}^1 \setminus \{1\})^{2p-k} \times \mathbb{A}^p \longrightarrow \mathcal{M}_{0,n} \times (\mathbb{P}^1 \setminus \{1\})^{2p-k}$$

restricted to Z is dominant, flat and equidimensional of dimension 0 (that is quasi-finite).

The symmetric group $\Sigma_{2p-k} \rtimes (\mathbb{Z}/2\mathbb{Z})^{2p-k}$ acts on $V_n^k(p)$ by permutation of the $\mathbb{P}^1 \setminus \{1\}$ factors and inversion $t_i \mapsto 1/t_i$ on the same factors. Let Alt_{2p-k} be the corresponding alternating projection. The symmetric group Σ_p acts on $V_n^k(p)$ by permutation of the \mathbb{A}^1 factors; let Sym_p denotes the corresponding symmetric projection.

The vector space $\mathcal{N}_n^k(p)$ is defined as $\text{Sym}_p \circ \text{Alt}_{2p-k}(V_n^k(p))$. For fixed p they form a complex with differential induced by intersection with faces of $(\mathbb{P}^1 \setminus \{1\})^{2p-k}$ given by $t_i = 0$ and $t_i = \infty$. Concatenation of factors and induced pull-back by the diagonal

$$\Delta_n : \mathcal{M}_{0,n} \longrightarrow \mathcal{M}_{0,n} \times \mathcal{M}_{0,n}$$

induce a product structure on

$$\mathcal{N}_n = \bigoplus_p \left(\bigoplus_k \mathcal{N}_n^k(p) \right).$$

This endows \mathcal{N}_n with the structure of differential graded commutative (and associative) algebra for the cohomological degree k in superscript. Following M. Levine in [Lev10], we obtain:

Corollary 5.3. *Let H_n be Hopf algebra given by the H^0 of the (associative) bar construction of \mathcal{N}_n (see [BK94, Lev10, Sou14]). Then, there is an isomorphism*

$$G_n \simeq \text{Spec}(H_n)$$

where G_n is the tannakian group associated to $\text{MTM}_{/\text{Spec}(\mathbb{F})}(\mathcal{M}_{0,n})$.

Moreover in the exact sequence

$$0 \longrightarrow K_n \longrightarrow G_n \xrightarrow[\phi_n]{\substack{\text{---} \xrightarrow{\bar{x}_v^*} \text{---}}} G(\text{Spec}(\mathbb{F})) \longrightarrow 0$$

any choice of a tangential based point x_v^* in $P_{n,\infty}$, identifies K_n with Deligne-Goncharov motivic fundamental group $\pi_1^{\text{mot}}(\mathcal{M}_{0,n}, x_v)$. Moreover it defines an action of $G(\text{Spec}(\mathbb{F}))$ on K_n which is coming from the action over $\text{Spec}(\mathbb{Z})$.

Note that the same holds for G_{n_1,n_2} and that families of base points x_v can be chosen in a compatible way. We then obtain

Corollary 5.4. *The tannakian group $G_3 = G(\mathbb{Z})$ injects in $GT^{\text{mot}}(\text{Spec}(\mathbb{Z}))$.*

Proof. The diagram defining $GT^{\text{mot}}(\text{Spec}(\mathbb{Z}))$ gives a morphism

$$G(\mathbb{Z}) \longrightarrow GT^{\text{mot}}(\text{Spec}(\mathbb{Z})).$$

F. Brown's work in [Bro12] shows that the action of $G(\mathbb{Z})$ on K_4 is faithful which implies the injectivity of the above morphisms. \square

How to describe explicitly the Hopf algebra H_n in terms of algebraic cycles in \mathcal{N}_n ; hence generalizing the construction of [Sou12]; will be addressed in a future work.

6. SOME OPEN QUESTIONS

We present in this last question some reasonable but still open problems related to the present work. They are of different types. The first type concerns a finer understanding of (derived) groups K_n . The second type relates our construction to classical approaches to the Grothendieck-Teichmüller tower.

First of all, working with \mathbb{Z} coefficient over $\text{Spec}(\mathbb{Z})$ forces to consider the triangulated categories $\text{DMT}_{/\text{Spec}(\mathbb{Z}),\mathbb{Z}}(\mathcal{M}_{0,n})$ and affine derived group schemes $G_{\mathbb{Z},n}^\bullet$. The geometric part $K_{\mathbb{Z},n}^\bullet$ of $G_{\mathbb{Z},n}^\bullet$ has been defined as the kernel of the structure map

$$0 \longrightarrow K_{\mathbb{Z},n}^\bullet \longrightarrow G_{\mathbb{Z},n}^\bullet \xrightarrow[\phi_n]{} G^\bullet(\mathbb{Z}) \longrightarrow 0$$

Conjecture 2. The affine derived group scheme $K_{\mathbb{Z},n}^\bullet$ is an affine group scheme:

$$K_{\mathbb{Z},n}^\bullet = K_{\mathbb{Z},n} = \text{Spec}(R_n)$$

where R_n is a commutative Hopf algebra (not necessarily co-commutative) defined over \mathbb{Z} .

Concerning the embedding of codimension 1 boundary components $D \simeq \mathcal{M}_{0,n_1} \times \mathcal{M}_{0,n_2}$ we conjecture:

Conjecture 3. The induced morphism

$$K_{\mathbb{Z},n_1,n_2}^\bullet \xrightarrow[p_{n_1,n_2}]{} K_{\mathbb{Z},n_1}^\bullet \times K_{\mathbb{Z},n_2}^\bullet$$

is an isomorphism.

A stronger version would state that

$$G_{\mathbb{Z},n_1,n_2}^\bullet \xrightarrow[p_{n_1,n_2}]{} G_{\mathbb{Z},n_1}^\bullet \times G_{\mathbb{Z},n_2}^\bullet$$

is an isomorphism. Conjectures 2 and 3 would allow to consider only group automorphisms of K_n^\bullet in the definition of $GT_{\mathbb{Z}}^\bullet(S)$.

As we have already seen, choices tangential bases points x_v in $P_{n,\infty}$ endows K_n with action of $G(\mathbb{Q})$ giving it a motivic structure. As $GT^{mot}(\mathrm{Spec}(\mathbb{F}))$ acts on the tower of the K_n , it acts on the tower of their realization. One expects:

Conjecture 4. One has the following isomorphism

$$GT^{mot}(\mathrm{Spec}(\mathbb{F})) \simeq GT \simeq \mathrm{Aut}(\mathcal{R}eal_{\mathrm{Betti}}(K_*)).$$

The weight filtration on the K_n induces a weight filtration on $GT^{mot}(\mathrm{Spec}(\mathbb{F}))$ and we denote by $GRT^{mot}(\mathrm{Spec}(F))$ the induced sum of graded pieces. With these notation one has

$$GRT^{mot}(\mathrm{Spec}(\mathbb{F})) \simeq GRT \simeq \mathrm{Aut}(\mathcal{R}eal_{\mathrm{De Rham}}(K_*)).$$

Note that in the above formulas the second isomorphism is a consequence of the work of Bar-Natan [BN98] following Drinfel'd work [Dri91].

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